Alpha Star Generalized \( \omega \) - Closed Sets in Bitopological Spaces

By

Qays Hatem Imran

Al-Muthanna University, College of Education, Department of Mathematics
Al-Muthanna, Iraq

E-mail : alrubaye84@yahoo.com

Abstract:
The aim of this paper is to introduce the concepts of alpha star generalized \( \omega \) - closed sets, alpha star generalized \( \omega \) - open sets and study their basic properties in bitopological spaces.

Keywords: \( \tau_1 \tau_2 \) - alpha star generalized \( \omega \) - closed sets, \( \tau_i \tau_j \) - alpha star generalized \( \omega \) - open sets, \( \tau_i \tau_j \) - generalized \( \omega \) - closed sets.

1. Introduction:
Levine, [7] initiated the study of generalized closed sets in topological spaces in 1970. In 1963, J. C. Kelly, [2] defined: a set equipped with two topologies is called a bitopological space, denoted by \((X, \tau_1, \tau_2)\) where \((X, \tau_1)\) and \((X, \tau_2)\) are two topological spaces. Semi generalized closed sets and generalized semi closed sets are extended to bitopological settings by F. H. Khedr and H. S. Al-saadi, [1]. K. Chandrasekhar Rao and K. Kannan, [5,6] introduced the concepts of semi star generalized closed sets in bitopological spaces. Moreover, the concept of generalized closed sets were introduced in ideal bitopological spaces by Noiri and Rajesh [9]. In 1986, T. Fukutake, [8] generalized this notion to bitopological spaces and he defined a set \( A \) of a bitopological space \( X \) to be an \( ij \)-generalized closed set (briefly \( ij \)-g-closed) if \( j - cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-open in \( X, i, j = 1,2 \) and \( i \neq j \). For any subset \( A \subseteq X \), \( \tau_i \)-int\((A) \) and \( \tau_i \)-cl\((A) \) denote the interior and closure of a set \( A \) with respect to the topology \( \tau_i \), for \( i = 1,2 \). The closure and interior with respect to the topology \( \tau_i \) of \( B \) relative to \( A \) is written as \( \tau_i \)-cl\(_A\)\(B\) and \( \tau_i \)-int\(_A\)\(B\) respectively. A point \( x \in X \) is called a condensation point of \( A \) if for each \( U \in \tau \) with \( x \in U \), the set \( U \cap A \) is uncountable. \( A \) is called \( \omega \) - closed if it contains all its condensation points. The complement of an \( \omega \) - closed set is called \( \omega \) - open. It is well known that a subset \( A \) of a space \( (X, \tau) \) is \( \omega \) - open if and only if for each \( x \in A \), there exists \( U \in \tau \) such that \( x \in U \) and \( U \cap W \) is countable. The family of all \( \omega \) - open subsets of a space \( (X, \tau) \), by \( \tau_\omega \) or \( \omega \mathcal{O}(X) \), forms a topology on \( X \) finer than \( \tau \). The \( \omega \) - closure and \( \omega \) - interior with respect to the topology \( \tau_i \), that can be defined in a manner similar to \( \tau_i \)-cl\((A) \) and \( \tau_i \)-int\((A) \), respectively, will be denoted by \( \tau_i \)-cl\(_\omega\)\(A\) and \( \tau_i \)-int\(_\omega\)\(A\), respectively. \( A^c \) or \( X - A \) denotes the
complement of $A$ in $X$ unless explicitly stated. The aim of this communication is to introduce the concepts of $\tau_1\tau_2$ - alpha star generalized closed sets, $\tau_1\tau_2$ - alpha star generalized $\omega$ - closed sets, $\tau_1\tau_2$ - alpha star generalized $\omega$ - open sets and study their basic properties in bitopological spaces. We shall require the following known definitions.

**Definition 1.1:**

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called
(i) $\tau_1\tau_2$ - $\alpha$ - open [4] if
$$A \subseteq \tau_1 - \text{int}(\tau_2 - \text{cl}(\tau_1 - \text{int}(A))).$$
(ii) $\tau_1\tau_2$ - $\alpha$ - closed [4] if $X - A$ is $\tau_1\tau_2$ - $\alpha$ - open. 
Equivalently, a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2$ - $\alpha$ - closed if $\tau_2 - \text{cl}(\tau_1 - \text{int}(\tau_2 - \text{cl}(A))) \subseteq A$.

(iii) $\tau_1\tau_2$ - generalized closed (briefly $\tau_1\tau_2 - g$ - closed) [8] if $\tau_2 - \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$ - open in $X$,
(iv) $\tau_1\tau_2$ - generalized open (briefly $\tau_1\tau_2 - g$ - open) [8] if $X - A$ is $\tau_1\tau_2 - g$ - closed.
(v) $\tau_1\tau_2$ - $\alpha$ generalized closed (briefly $\tau_1\tau_2 - \alpha g$ - closed) [4] if $\tau_2 - \alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$ - open in $X$. 
(vi) $\tau_1\tau_2$ - $\alpha$ generalized open (briefly $\tau_1\tau_2 - \alpha g$ - open) [4] if $X - A$ is $\tau_1\tau_2 - \alpha g$ - closed.

**Definition 1.2:**

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called
(i) $\tau_1\tau_2$ - generalized $\omega$ - closed (briefly $\tau_1\tau_2 - g\omega$ - closed) [3] if $\tau_2 - \text{cl}_\omega(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$ - open in $X$.
(ii) $\tau_1\tau_2$ - generalized $\omega$ - open (briefly $\tau_1\tau_2 - g\omega$ - open) [3] if $X - A$ is $\tau_1\tau_2 - g\omega$ - closed.
(iii) $\tau_1\tau_2$ - $\alpha$ generalized $\omega$ - closed (briefly $\tau_1\tau_2 - \alpha g\omega$ - closed) if $\tau_2 - \alpha\text{cl}_\omega(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$ - open in $X$.
(iv) $\tau_1\tau_2$ - $\alpha$ generalized $\omega$ - open (briefly $\tau_1\tau_2 - \alpha g\omega$ - open) if $X - A$ is $\tau_1\tau_2 - \alpha g\omega$ - closed.

2. Alpha Star Generalized Closed Sets:

In this section we define and study the concept of $\tau_1\tau_2 - \alpha^*$ generalized closed sets in bitopological spaces.

**Definition 2.1:**

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2 - \alpha^*$ generalized closed (briefly $\tau_1\tau_2 - \alpha^* g$ - closed) if $\tau_2 - \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$ - open in $X$.

**Example 2.2:**

Let $X = \{a, b, c\}, \tau_1 = \{\emptyset, X, \{a\}, \{b, c\}\}$, \(\tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}\). Then $\{a, b\}$ is $\tau_1\tau_2 - \alpha^* g$ - closed and $\{a\}$ is not $\tau_1\tau_2 - \alpha^* g$ - closed.

**Definition 2.3:**

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2 - \alpha^*$ generalized open (briefly $\tau_1\tau_2 - \alpha^* g$ - open) if and only if $X - A$ is $\tau_1\tau_2 - \alpha^* g$ - closed.
Theorem 2.4:

The arbitrary union of $\tau_i \tau_2 - \alpha^* g$ - closed sets $A_i, i \in I$ in a bitopological space $(X, \tau_1, \tau_2)$ is $\tau_i \tau_2 - \alpha^* g$ - closed if the family \{ $A_i, i \in I$ \} is $\tau_2$ - locally finite.

Proof:

Let \{ $A_i, i \in I$ \} be $\tau_2$ - locally finite and $A_i$ is $\tau_i \tau_2 - \alpha^* g$ - closed in $X$ for each $i \in I$. Let $\bigcup A_i \subseteq U$ and $U$ is $\tau_1$ - open in $X$. Then, $A_i \subseteq U$ and $U$ is $\tau_1$ - open in $X$ for each $i \in I$. Since $A_i$ is $\tau_i \tau_2 - \alpha^* g$ - closed in $X$ for each $i \in I$, we have $\tau_2 - cl(A_i) \subseteq U$.

Consequently, $\bigcup [\tau_2 - cl(A_i)] \subseteq U$. Since the family \{ $A_i, i \in I$ \} be $\tau_2$ - locally finite, $\tau_2 - cl(\bigcup(A_i)) = \bigcup (\tau_2 - cl(A_i)) \subseteq U$.

Therefore, $\bigcup A_i$ is $\tau_i \tau_2 - \alpha^* g$ - closed in $X$. ■

Theorem 2.5:

The arbitrary intersection of $\tau_i \tau_2 - \alpha^* g$ - open sets $A_i, i \in I$ in a bitopological space $(X, \tau_1, \tau_2)$ is $\tau_i \tau_2 - \alpha^* g$ - open if the family \{ $A_i^c, i \in I$ \} is $\tau_2$ - locally finite.

Proof:

Let \{ $A_i^c, i \in I$ \} be $\tau_2$ - locally finite and $A_i$ is $\tau_i \tau_2 - \alpha^* g$ - open in $X$ for each $i \in I$. Then, $A_i^c$ is $\tau_i \tau_2 - \alpha^* g$ - closed in $X$ for each $i \in I$. Then by theorem (2.4), we have $\bigcup (A_i^c) = \tau_i \tau_2 - \alpha^* g$ - closed in $X$.

Consequently, $(\bigcap A_i)^c$ is $\tau_i \tau_2 - \alpha^* g$ - closed in $X$. Therefore, $\bigcap A_i$ is $\tau_i \tau_2 - \alpha^* g$ - open in $X$. ■

3. Alpha Star Generalized $\omega$ - Closed Sets:

In this section we define and study the concept of $\tau_i \tau_2 - \alpha^*$ generalized $\omega$ - closed sets in bitopological spaces.

Definition 3.1:

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_i \tau_2 - \alpha^*$ generalized $\omega$ - closed (briefly $\tau_i \tau_2 - \alpha^* g\omega$ - closed) if $\tau_2 - cl_{\omega}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$ - open in $X$.

Example 3.2:

Let $X$ be the set of all real numbers $R$, $\tau_1 = \{ \phi, R, R - Q \}$, $\tau_2 = \{ \phi, R, Q \}$, where $Q$ is the set of all rational numbers. Then $R - Q$ is $\tau_i \tau_2 - \alpha^* g\omega$ - closed.

Theorem 3.3:

Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A \subseteq X$. Then the following are true.

(i) If $A$ is $\tau_2 - \omega$ - closed, then $A$ is $\tau_i \tau_2 - \alpha^* g\omega$ - closed.

(ii) If $A$ is $\tau_1$ - open and $\tau_i \tau_2 - \alpha^* g\omega$ - closed, then $A$ is $\tau_2 - \omega$ - closed.

(iii) If $A$ is $\tau_i \tau_2 - \alpha^* g\omega$ - closed, then $A$ is $\tau_i \tau_2 - g\omega$ - closed.

Proof:

(i) Suppose that $A$ is $\tau_2 - \omega$ - closed, let $A \subseteq U$ and $U$ is $\tau_1$ - open in $X$. Then $\tau_2 - cl_{\omega}(A) = A \subseteq U$. Consequently, $A$ is $\tau_i \tau_2 - \alpha^* g\omega$ - closed.

(ii) Suppose that $A$ is $\tau_1$ - open and $\tau_i \tau_2 - \alpha^* g\omega$ - closed. Let $A \subseteq A$ and $A$ is $\tau_1$ - open. Then $\tau_2 - cl_{\omega}(A) \subseteq A$. Therefore,
\[ \tau_2 - cl_\omega(A) = A. \quad \text{Consequently} \quad A \text{ is } \tau_2 - \omega \text{-closed.} \]

(iii) Suppose that \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed} \), let \( A \subseteq U \) and \( U \) is \( \tau_1 \text{-open in } X \). Since \( U \) is \( \tau_1 \text{-open in } X \), we have \( \tau_2 - cl_\omega(A) \subseteq U \).

Consequently, \( A \) is \( \tau_1 \tau_2 - g \omega \text{-closed}. \]

Since, \( \tau_2 - cl_\omega(A) \subseteq \tau_2 - cl(A) \), we have the following theorem.

**Theorem 3.4:**

Every \( \tau_1 \tau_2 - \alpha^* g \) -closed set is \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed} \) and every \( \tau_2 \) -closed set is \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed} \).

**Remark 3.5:**

From the theorem (3.3), theorem (3.4) and above definitions, we have the following relations.

**Theorem 3.6:**

If \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed in } X \) and \( A \subseteq B \subseteq \tau_2 - cl_\omega(A) \), then \( B \) is \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed}. \]

**Proof:**

Suppose that \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed in } X \) and \( A \subseteq B \subseteq \tau_2 - cl_\omega(A) \). Let \( B \subseteq U \) and \( U \) is \( \tau_1 \text{-open in } X \). Then \( A \subseteq U \). Since \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed} \), we have \( \tau_2 - cl_\omega(A) \subseteq U \). Since \( B \subseteq \tau_2 - cl_\omega(A) \), \( \tau_2 - cl_\omega(B) \subseteq \tau_2 - cl_\omega(A) \subseteq U \). Hence \( B \) is \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed}. \]

**Theorem 3.7:**

If \( A \) and \( B \) are \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed sets} \) then so is \( A \cup B \).

**Proof:**

Suppose that \( A \) and \( B \) are \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed sets} \). Let \( U \) be \( \tau_1 \text{-open in } X \) and \( A \cup B \subseteq U \). Then \( A \subseteq U \) and \( B \subseteq U \). Since \( A \) and \( B \) are \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed sets} \), we have \( \tau_2 - cl_\omega(A) \subseteq U \) and \( \tau_2 - cl_\omega(B) \subseteq U \).

Consequently, \( \tau_2 - cl_\omega(A \cup B) \subseteq U \).

Therefore, \( A \cup B \) is \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed}. \]

**Theorem 3.8:**

Let \( B \subseteq A \subseteq X \) where \( A \) is \( \tau_1 \text{-open} \) and \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed} \) in \( X \). Then \( B \) is \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed} \) relative to \( A \) if and only if \( B \) is \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed} \) relative to \( X \).

**Proof:**

Suppose that \( B \subseteq A \subseteq X \) where \( A \) is \( \tau_1 \text{-open} \) and \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed} \). Suppose that \( B \) is \( \tau_1 \tau_2 - \alpha^* g \omega \text{-closed} \) relative to \( A \).

Let \( B \subseteq U \) and \( U \) is \( \tau_1 \text{-open in } X \). Since
A ⊆ X , A is τ₁ - open, we have A ∪ U is τ₁ - open in X . Consequently A ∩ U is τ₁ - open in A . Since B ⊆ A , B ⊆ U , we have B ⊆ A ∩ U . Since B is τ₁ τ₂ - α⁺ gω - closed relative to A , we have τ₂ - clω(Bₐ) ⊆ A ∩ U . Therefore , τ₂ - clω(Bₐ) ⊆ U .

Since A is τ₁ - open , we have τ₂ - clω(Bₐ) = τ₂ - clω(B) ∩ A = τ₂ - clω(B) ⊆ U . Hence B is τ₁ τ₂ - α⁺ gω - closed relative to X .

Conversely, suppose that B is τ₁ τ₂ - α⁺ gω - closed relative to X . Let B ⊆ U and U is τ₁ - open in A . Since A ⊆ X , we have U is τ₁ - open in X . Since B is τ₁ τ₂ - α⁺ gω - closed relative to X , we have τ₂ - clω(B) ⊆ U . Now , τ₂ - clω(Bₐ) = τ₂ - clω(B) ∩ A = τ₂ - clω(B) ⊆ U . Therefore , B is τ₁ τ₂ - α⁺ gω - closed relative to A .

**Corollary 3.9:**

If A is τ₁ τ₂ - α⁺ gω - closed , τ₁ - open in X and F is τ₂ - ω - closed in X , then A ∩ F is τ₂ - ω - closed in X .

**Proof:**

Since A is τ₁ τ₂ - α⁺ gω - closed , τ₁ - open in X , we have A is τ₂ - ω - closed in X . { By Theorem (3.3) (ii) }. Since F is τ₂ - ω - closed in X , A ∩ F is τ₂ - ω - closed in X .

**Theorem 3.10:**

If A is τ₁ τ₂ - α⁺ gω - closed in X , then τ₂ - clω(A) - A contains no nonempty τ₁ - closed set .

**Proof:**

Suppose that A is τ₁ τ₂ - α⁺ gω - closed in X . Let F be τ₁ - closed and

\[ F \subseteq \tau₂ - clω(A) - A . \]

Since F be τ₁ - closed , we have F^c is τ₁ - open .

Since F ⊆ τ₂ - clω(A) - A , we have

\[ F \subseteq \tau₂ - clω(A) \]

and A ⊆ F^c . Since A is τ₁ τ₂ - α⁺ gω - closed in X , we have

\[ \tau₂ - clω(A) \subseteq F^c . \]

Consequently, F = φ . Hence τ₂ - clω(A) - A contains no nonempty τ₁ - closed set .

**Corollary 3.11:**

Let A be τ₁ τ₂ - α⁺ gω - closed , then A is τ₂ - ω - closed if and only if τ₂ - clω(A) - A is τ₁ - closed .

**Proof:**

Suppose that A is τ₁ τ₂ - α⁺ gω - closed . Since A is τ₂ - ω - closed , we have τ₂ - clω(A) = A . Then τ₂ - clω(A) - A = φ is τ₁ - closed . Conversely, suppose that A is τ₁ τ₂ - α⁺ gω - closed and τ₂ - clω(A) - A is τ₁ - closed . Since A is τ₁ τ₂ - α⁺ gω - closed , we have τ₂ - clω(A) - A contains no nonempty τ₁ - closed set { by Theorem (3.10) }. Since τ₂ - clω(A) - A is itself τ₁ - closed , we have τ₂ - clω(A) - A = φ . Then τ₂ - clω(A) = A . Hence A is τ₂ - ω - closed .

**Theorem 3.12:**

If A is τ₁ τ₂ - α⁺ gω - closed and

A ⊆ B ⊆ τ₂ - clω(A) , then τ₂ - clω(B) - B contains no nonempty τ₁ - closed set .

**Proof:**

Let A be τ₁ τ₂ - α⁺ gω - closed and
A \subseteq B \subseteq \tau_2 - cl_\omega(A)$. Then $B$ is
\(\tau_1 \tau_2 - \alpha^* g\omega\)-closed. (By Theorem (3.6)).
Hence $\tau_2 - cl_\omega(B) - B$ contains no nonempty $\tau_1$-closed set. (By Theorem (3.10)). 

4. Alpha Star Generalized $\omega$ - Open Sets:

We begin this section with a relatively new definition.

Definition 4.1:

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1 \tau_2 - \alpha^*$ generalized $\omega$-open (briefly $\tau_1 \tau_2 - \alpha^* g\omega$-open) if and only if $X - A$ is $\tau_1 \tau_2 - \alpha^* g\omega$-closed.

Example 4.2:

In Example (3.2), $Q$ is $\tau_1 \tau_2 - \alpha^* g\omega$-open.

Theorem 4.3:

A set $A$ is $\tau_1 \tau_2 - \alpha^* g\omega$-open if and only if $F \subseteq \tau_2 - \text{int}_\omega(A)$ whenever $F$ is $\tau_1$-closed and $F \subseteq A$.

Proof:

Suppose that $A$ is $\tau_1 \tau_2 - \alpha^* g\omega$-open. Suppose that $F$ is $\tau_1$-closed and $F \subseteq A$. Then $F^c$ is $\tau_1$-open and $A^c \subseteq F^c$. Since $A^c$ is $\tau_1 \tau_2 - \alpha^* g\omega$-closed, we have $\tau_2 - cl_\omega(A^c) \subseteq F^c$. Since $\tau_2 - cl_\omega(A^c) = (\tau_2 - \text{int}_\omega(A))^c$, we have $F \subseteq \tau_2 - \text{int}_\omega(A)$.

Conversely, suppose that $F \subseteq \tau_2 - \text{int}_\omega(A)$ whenever $F$ is $\tau_1$-closed and $F \subseteq A$. Then $A^c \subseteq F^c$ and $F^c$ is $\tau_1$-open. Since $F \subseteq \tau_2 - \text{int}_\omega(A)$, and $\tau_2 - cl_\omega(A^c) = (\tau_2 - \text{int}_\omega(A))^c$, we have $\tau_2 - cl_\omega(A^c) \subseteq U$.

Then $A^c$ is $\tau_1 \tau_2 - \alpha^* g\omega$-closed. Consequently, $A$ is $\tau_1 \tau_2 - \alpha^* g\omega$-open.

Theorem 4.4:

If $A$ and $B$ are separated $\tau_1 \tau_2 - \alpha^* g\omega$-open sets then $A \cup B$ is $\tau_1 \tau_2 - \alpha^* g\omega$-open set.

Proof:

Suppose $A$ and $B$ are separated $\tau_1 \tau_2 - \alpha^* g\omega$-open sets. Let $F$ be $\tau_1$-closed and $F \subseteq A \cup B$. Since $A$ and $B$ are separated , we have $\tau_1 - cl(A) \cap B = A \cap \tau_1 - cl(B) = \emptyset$ and $\tau_2 - cl(A) \cap B = A \cap \tau_2 - cl(B) = \emptyset$.

Then, $F \cap \tau_2 - cl(A) \subseteq (A \cup B) \cap \tau_2 - cl(A) = A$. Similarly, we can prove $F \cap \tau_2 - cl(B) \subseteq B$. Since $F$ is $\tau_1$-closed, we have $F \cap \tau_1 - cl(A)$ and $F \cap \tau_1 - cl(B)$ are $\tau_1$-closed. Since $A$ and $B$ are $\tau_1 \tau_2 - \alpha^* g\omega$-open, we have $F \cap \tau_2 - cl(A) \subseteq \tau_2 - \text{int}_\omega(A)$ and $F \cap \tau_2 - cl(B) \subseteq \tau_2 - \text{int}_\omega(B)$. Now,

$F = F \cap (A \cup B) \subseteq [F \cap \tau_2 - cl(A)] \cup [F \cap \tau_2 - cl(B)] \subseteq \tau_2 - \text{int}_\omega(A \cup B)$.

Therefore, $A \cup B$ is $\tau_1 \tau_2 - \alpha^* g\omega$-open.

Theorem 4.5:

If $A$ and $B$ are $\tau_1 \tau_2 - \alpha^* g\omega$-open sets then so is $A \cap B$.

Proof:

Suppose that $A$ and $B$ are $\tau_1 \tau_2 - \alpha^* g\omega$-open sets. Let $F$ be $\tau_1$-closed and $F \subseteq A \cap B$. Then, $F \subseteq A$ and $F \subseteq B$. Since $A$ and $B$ are $\tau_1 \tau_2 - \alpha^* g\omega$-open, we have $F \subseteq \tau_2 - \text{int}_\omega(A)$ and $F \subseteq \tau_2 - \text{int}_\omega(B)$.

Hence $F \subseteq \tau_2 - \text{int}_\omega(A \cap B)$.
Consequently, \( A \cap B \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open set.

**Theorem 4.6:**

If \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open in \( X \) and \( \tau_2 - \text{int}_{\omega} (A) \subseteq B \subseteq A \), then \( B \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open.

**Proof:**

Suppose that \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open in \( X \) and \( \tau_2 - \text{int}_{\omega} (A) \subseteq B \subseteq A \). Let \( F \) be \( \tau_1 \)-closed and \( F \subseteq B \). Since \( F \subseteq B \), \( B \subseteq A \), we have \( F \subseteq A \). Since \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open, we have

\[ F \subseteq \tau_2 - \text{int}_{\omega} (A). \]

Since \( \tau_2 - \text{int}_{\omega} (A) \subseteq B \), we have \( \tau_2 - \text{int}_{\omega} (A) \subseteq \tau_2 - \text{int}_{\omega} (B) \).

Then \( F \subseteq \tau_2 - \text{int}_{\omega} (B) \). Therefore, \( B \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open set.

**Theorem 4.7:**

If \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-closed in \( X \) then \( \tau_2 - \text{cl}_{\omega} (A) - A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open.

**Proof:**

Suppose that \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-closed in \( X \). Let \( F \) be \( \tau_1 \)-closed and

\[ F \subseteq \tau_2 - \text{cl}_{\omega} (A) - A. \]

Since \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-closed in \( X \), \( \tau_2 - \text{cl}_{\omega} (A) - A \) contains no nonempty \( \tau_1 \)-closed set. Since

\[ F \subseteq \tau_2 - \text{cl}_{\omega} (A) - A, \quad F = \phi \subseteq \tau_2 - \text{int}_{\omega} (\tau_2 - \text{cl}_{\omega} (A) - A). \]

Therefore,

\[ \tau_2 - \text{cl}_{\omega} (A) - A \] is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open.

**Theorem 4.8:**

If \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open in a bitopological space \( (X, \tau_1, \tau_2) \), then \( G = X \) whenever \( G \) is \( \tau_1 \)-open and \( \tau_2 - \text{cl}_{\omega} (A) \cup A^c \subseteq G \).

**Proof:**

Suppose that \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open in a bitopological space \( (X, \tau_1, \tau_2) \) and \( G \) is \( \tau_1 \)-open and \( \tau_2 - \text{cl}_{\omega} (A) \cup A^c \subseteq G \).

Then, \( G^c \subseteq (\tau_2 - \text{int}_{\omega} (A) \cup A^c)^c \)

\[ = \tau_2 - \text{cl}_{\omega} (A^c) - A^c. \]

Since \( G^c \) is \( \tau_1 \)-closed and \( A^c \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-closed, we have

\[ \tau_2 - \text{cl}_{\omega} (A^c) - A^c \] contains no nonempty \( \tau_1 \)-closed set in \( X \) (By Theorem (3.10)).

Therefore, \( G^c = \phi \). Hence \( G = X \).

**Theorem 4.9:**

The intersection of a \( \tau_1 \tau_2 - \alpha^* g \omega \)-open set and \( \tau_1 - \omega \)-open set is always \( \tau_1 \tau_2 - \alpha^* g \omega \)-open.

**Proof:**

Suppose that \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open and \( B \) is \( \tau_1 - \omega \)-open. Then \( B^c \) is \( \tau_2 - \omega \)-closed. Therefore, \( B^c \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-closed. (By Theorem (3.3) (i)). Hence \( B \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open. Consequently, \( A \cap B \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open. (By Theorem (4.5)).

**Theorem 4.10:**

If \( A \times B \) is \( \tau_1 \times \sigma_1 \tau_2 \times \sigma_2 - \alpha^* g \omega \)-open subset of \((X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)\), then \( A \) is \( \tau_1 \tau_2 - \alpha^* g \omega \)-open subset in \((X, \tau_1, \tau_2)\) and \( B \) is \( \sigma_1 \sigma_2 - \alpha^* g \omega \)-open subset in \((Y, \sigma_1, \sigma_2)\).

**Proof:**

Let \( F \) be a \( \tau_1 \)-closed subset of \((X, \tau_1, \tau_2)\) and let \( G \) be a \( \sigma_1 \)-closed subset.
of \((Y, \sigma_1, \sigma_2)\) such that \(F \subseteq A\) and \(G \subseteq B\). Then \(F \times G\) is \(\tau_1 \times \sigma_1\) - closed in 
\((X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)\) such that
\(F \times G \subseteq A \times B\). By assumption \(A \times B\) is 
\(\tau_1 \times \sigma_1 \tau_2 \times \sigma_2 - \alpha^* g \omega\) - open in
\((X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)\) and so \(F \times G \subseteq 
\tau_2 \times \sigma_2 - \text{int}_\omega(A \times B) \subseteq \tau_2 - \text{int}_\omega(A) \times 
\sigma_2 - \text{int}_\omega(B)\). Therefore \(F \subseteq \tau_2 - \text{int}_\omega(A)\) and
\(G \subseteq \sigma_2 - \text{int}_\omega(B)\). Hence \(A\) is
\(\tau_1 \tau_2 - \alpha^* g \omega\) - open and \(B\) is \(\sigma_1 \sigma_2 - \alpha^* g \omega\) - open. ■

References:

المستخلص:
الهدف من هذا البحث هو تقديم مفاهيم مجموعات الفا ستار المعممة \(\omega\) - المغلقة ، مجموعات الفا ستار المعممة \(\omega\) - المفتوحة ودراسة خصائصها الأساسية في الفضاءات ثنائية التبولوجي.