Ministry of Higher Education
& Scientific Research
Muthanna University
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Department of Mathematics

Lectures in Calculus

For first class

By

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FUNCTIONS

Functions: Functions are a tool for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description.

$$y = f(x) \text{ ("}y\text{ equals } f \text{ of } x\text{").}$$

DEFINITION : A function f from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

Domain and Range

Domain: The set D of all possible input values is called the **domain** of the function.
Range: The set of all values of $f(x)$ as x varies throughout D is called the **range** of the function.
Remarks:
1 – The symbol f represents the function, the letter x is the **independent variable** representing the input value of f, and y is the **dependent variable** or output value of f at x.
2 – The range may not include every element in the set Y.
3 – The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line.
4 – Often a function is given by a formula that describes how to calculate the output value from the input variable.
5 – When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x-values for which the formula gives real y-values, the so-called **natural domain**.
6 – Changing the domain to which we apply a formula usually changes the range as well.
7 – When the range of a function is a set of real numbers, the function is said to be **real-valued**.
8 – The domains and ranges of many real-valued functions of a real variable are intervals or combinations of intervals.
9 – The interva1s may be open, closed, or half open, and may be finite or infinite.
10 – The range of a function is not always easy to find.
11 – A function f is like a machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Figure 1.1).

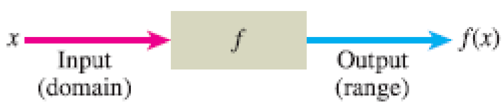


FIGURE 1.1 A diagram showing a function as a kind of machine.

12 – A function can also be pictured as an arrow diagram (Figure 1.2).

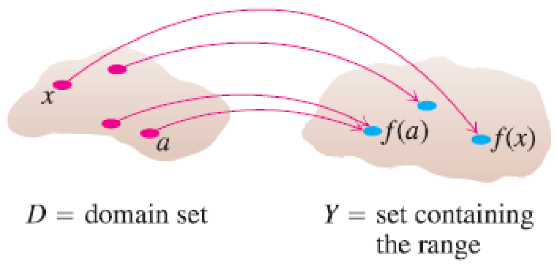


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D.

13 – A function can have the same *value* at two different input elements in the domain, but each input element x is assigned a *single* output value *f(x).*

EXAMPLE 1: Let's verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	**Domain (x)**	**Range (y)**
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution: 1 – The formula $y = x^2$ gives a real y-value for any real number x, so the domain is (- ∞, ∞). The range of $y = x^2$ is [0, ∞) because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, $y = \left(\sqrt{y}\right)^2$ for $y \geq 0$.

2 – The formula y = l/x gives a real y-value for every x except x = 0. For consistency in the rules of arithmetic, we *cannot divide any number by zero.* The range of y = 1/x , the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since y = 1/(1/y) . That is, for $y \neq 0$ the number x = 1/y is the input assigned to the output value y.

3 – The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is [0, ∞) because every nonnegative number is some number's square root (namely, it is the square root of its own square).

4 – In $y = \sqrt{4 - x}$, the quantity 4 – x cannot be negative . That is, 4 – $x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is [0, ∞), the set of all nonnegative numbers.

5 – The formula $y = \sqrt{1 - x^2}$ gives a real *y-value* for every x in the closed interval from - 1 to 1. Outside this domain. $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is [0, 1].

Graphs of Functions:

If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input - output pairs for f. In set notation. The graph is $\{(x, f(x)): x \in D\}$.

EXAMPLE: Graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y).

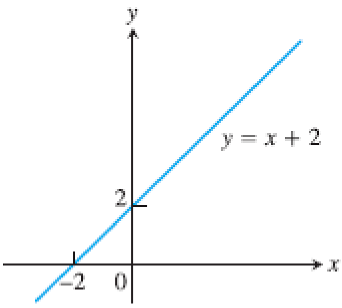


FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

EXAMPLE 2: Graph the function $y = x^2$ over the interval [-2, 2].

Solution: Make a table of xy-pairs that satisfy the equation $y = x^2$. Plot the points (x, y) whose coordinates appear in the table, and draw a *smooth* curve (labeled with its equation) through the plotted points (see Figure 1.4).

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

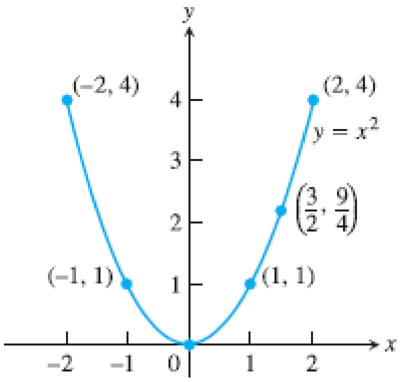


FIGURE 1.4 Graph of the function in Example 2.

Examples for Graphs the function:

1 - Representing a Function Numerically

We have seen how a function may be represented algebraically by a formula (the area function) and visually by a graph. Another way to represent a function is numerically, the graph consisting of only the points in the table is called a scatterplot.

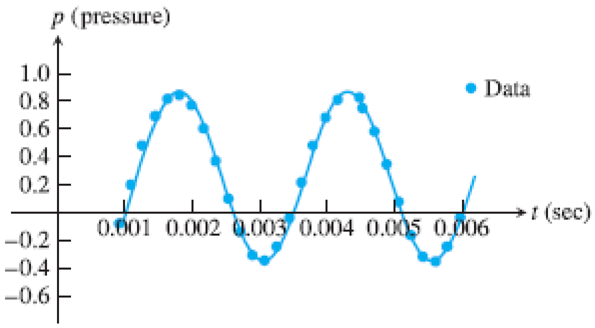


FIGURE 1.5 A smooth curve through the plotted points gives a graph of the pressure function

2 - The Vertical Line Test for a Function

Not every curve in the coordinate plane can be the graph of a function. A function *f* can have only one value *f(x)* for each *x* in its domain, so *no vertical* line can intersect the graph of a function more than once. If *a* is in the domain of the function *f*, then the vertical line *x* = *a* will intersect the graph of *f* at the single point *(a, f(a))* .

A circle cannot be the graph of a function since some vertical lines intersect the circle twice. The circle in Figure 1.6a, however, does contain the graphs of *two* functions of *x:* The upper semicircle defined by the function *f(x)* $=\sqrt{1-x^2}$ and the lower semicircle defined by the function *g(x)* = - $\sqrt{1-x^2}$ (Figures 1.6b and 1.6c).

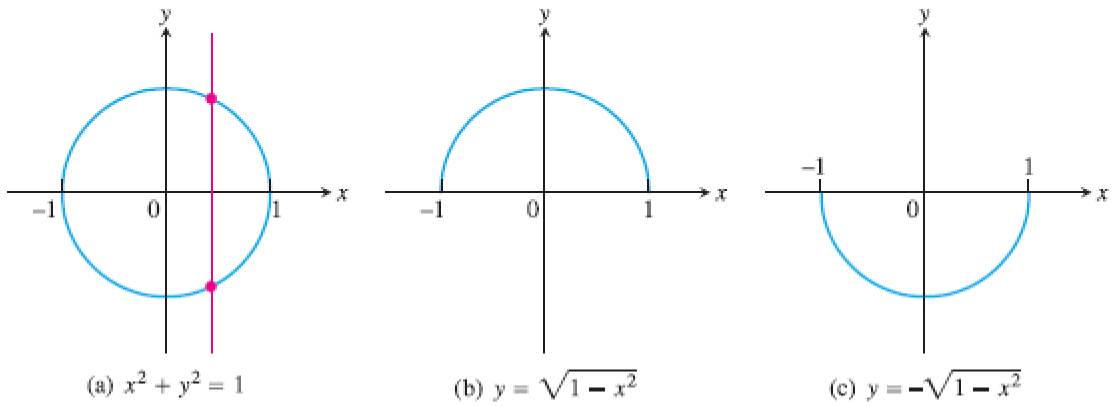


FIGURE 1.6 (a) The circle is not the graph of a function; it fails the vertical line test . (b) The upper semicircle is the graph of a function *f(x)* $= \sqrt{1-x^2}$. (c) The lower semicircle is the graph of a function g(x) = $-\sqrt{1-x^2}$.

3 - Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

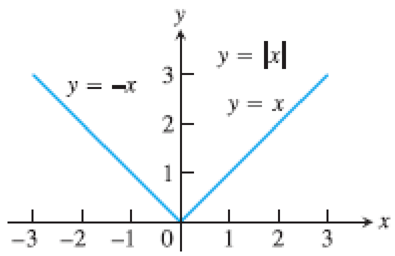


FIGURE 1.7 The absolute value function has domain (- ∞, ∞) and range [0, ∞).

EXAMPLE 3: The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

is defined on the entire real line but has values given by different formulas depending on the position of *x*. The values of *f* are given by *y* = -*x* when *x* < 0, $y = x^2$ when $0 \le x \le 1$, and *y* = 1 when *x* > 1. The function, however, is *just one function* whose domain is the entire set of real numbers (Figure 1.8).

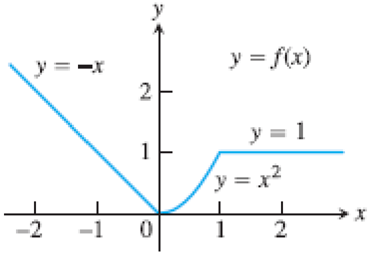


FIGURE 1.8 To graph the function *y* = *f(x)* shown here, we apply different formulas to different parts of its domain (Example 3).

EXAMPLE 4: The function whose value at any number *x* is the *greatest integer less than or equal to x* is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$. Figure 1.9 shows the graph. Observe that

$\lfloor 2.4 \rfloor = 2$, $\lfloor 1.9 \rfloor = 1$, $\lfloor 0 \rfloor = 0$, $\lfloor -1.2 \rfloor = 2$, $\lfloor 2 \rfloor = 2$, $\lfloor 0.2 \rfloor = 0$,$\lfloor -0.3 \rfloor = -1$, $\lfloor -2 \rfloor = -2$

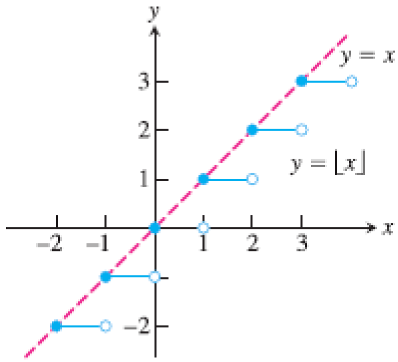


FIGURE 1.9 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line *y* = *x*, so it provides an integer floor for *x* (Example 4).

EXAMPLE 5: The function whose value at any number *x* is the *smallest integer greater than or equal to x* is called the **least integer function** or the **integer ceiling function** . It is denoted $\lceil x \rceil$. Figure 1.10 shows the graph. For positive values of *x*, this function might represent.

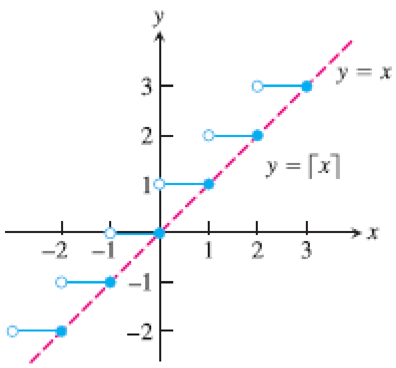


FIGURE 1.10 The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line *y* = *x* , so it provides an integer ceiling for *x* (Example 5) .

4 - Increasing and Decreasing Functions

If the graph of a function *climbs* or *rises* as you move from left to right, we say that the function is *increasing*. If the graph *descends* or *falls* as you move from left to right, the function is *decreasing*.

> **DEFINITIONS** Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.
>
> **1.** If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
>
> **2.** If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for *every* pair of points x_1 and x_2 in I with $x_1 < x_2$. Because we use the inequality $<$ to compare the function values, instead of $\leq$, it is sometimes said that f is *strictly* increasing or decreasing on I. The interval I may be finite (also called bounded) or infinite (unbounded) and by definition never consists of a single point.

EXAMPLE 7: The function graphed in Figure 1.8 is decreasing on $(-\infty, 0]$ and increasing on $[0, 1]$. The function is neither increasing nor decreasing on the interval $[1, \infty)$ because of the strict inequalities used to compare the function values in the definitions.

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have characteristic symmetry properties.

> **DEFINITIONS** A function $y = f(x)$ is an
>
> **even function of x** if $f(-x) = f(x)$,
>
> **odd function of x** if $f(-x) = -f(x)$,
>
> for every x in the function's domain.

The graph of an even function is **symmetric about they-axis**. Since $f(-x) = f(x)$.

The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$.

FIGURE 1.11 (a) The graph of $y = x^2$ (an even function) is symmetric about the y-axis. (b) The graph *of* $y = x^3$ (an odd function) is symmetric about the origin.

EXAMPLE 8

$f(x) = x^2$ Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis.

$f(x) = x^2 + 1$ Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis. (Figure1.12a).

$f(x) = x$ Odd function: $(-x) = -x$ for all x; symmetry about the origin.

$f(x) = x + 1$ Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal. Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.12b).

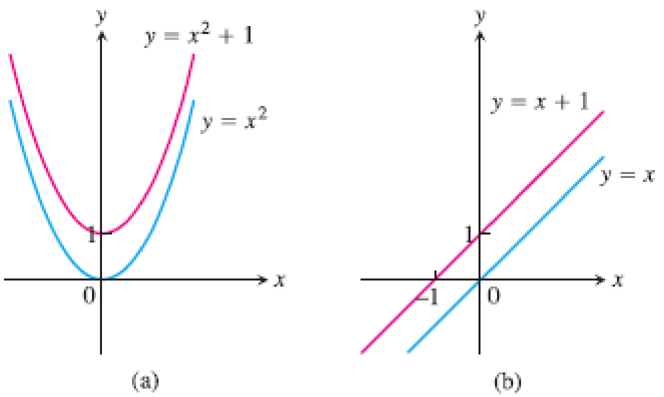


FIGURE 1.12 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd. The symmetry about the origin is lost.

Common Functions:

A variety of important types of functions are frequently encountered in calculus. We identify and briefly describe them here.

Linear Functions A function of the form $f(x) = mx + b$, for constants m and b, is called a **linear function.** Figure 1.13a shows an array of lines $f(x) = mx$ where $b = 0$, so these lines pass through the origin. The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function.** Constant functions result when the slope $m = 0$ (Figure1.13b). A linear function with positive slope whose graph passes through the origin is called a *proportionality* relationship.

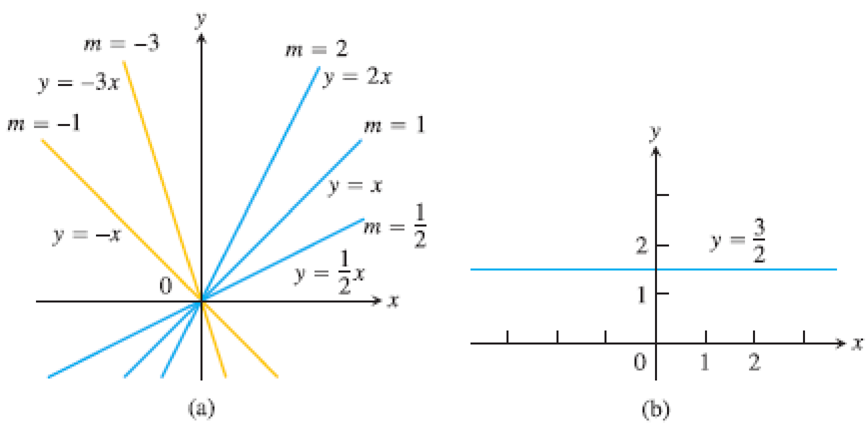


FIGURE 1.13 (a) Lines through the origin with slope m. (b) A constant function with slope $m = 0$.

DEFINITION Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other; that is, if $y = kx$ for some nonzero constant k.

If the variable y is proportional to the reciprocal 1/x, then sometimes it is said that y is **inversely proportional** to x (because 1/x is the multiplicative inverse of x).

Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function.** There are several important cases to consider.

(a) $a = n$, a positive integer.

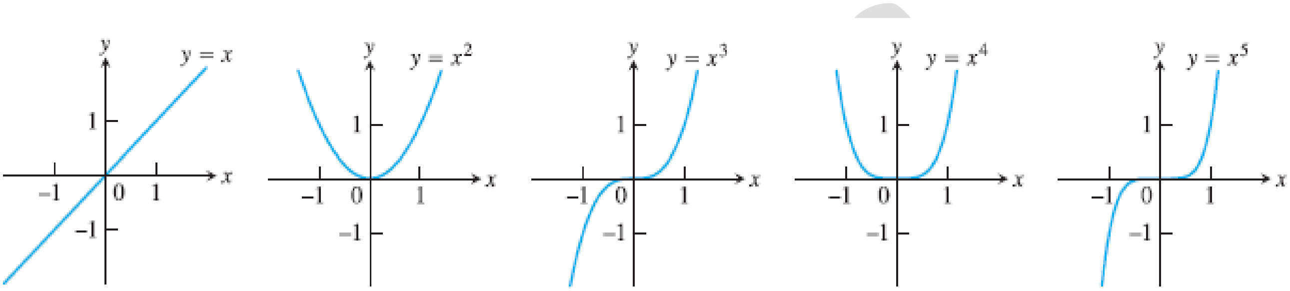


FIGURE 1.14 Graphs of $f(x) = x^n$, n = 1,2,3,4,5, defined for $-\infty < x < \infty$.

(b) $a = -1$ *or* $a = -2$.

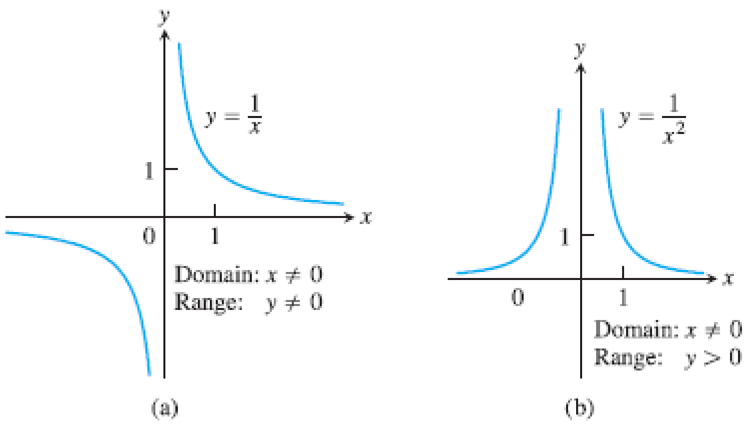


FIGURE 1.15 Graphs of the power functions $f(x) = x^a$ for part (a) a = -1 and for part (b) a = -2.

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$ *and* $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root and cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x. Their graphs are displayed in Figure 1.16 along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = \left(x^{1/2}\right)^3$ and $x^{2/3} = \left(x^{1/3}\right)^2$ **.**

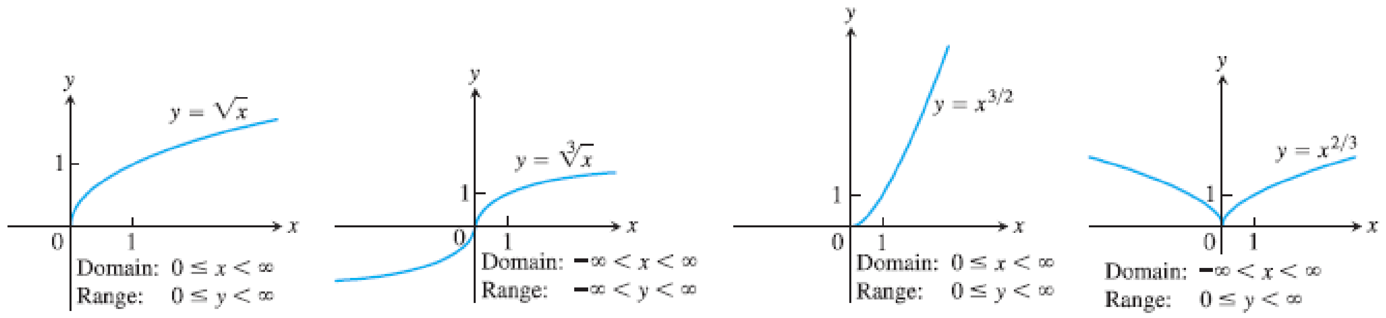


FIGURE 1.16 Graphs of the power functions $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$ *and* $\frac{2}{3}$.

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the **coefficients** of the polynomial) . All polynomials have domain (- ∞, ∞). If the leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the **degree** of the polynomial. Linear functions with m ≠ 0 are polynomials of degree 1. Polynomials of degree 2 , usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions .** Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3. Figure 1.17 shows the graphs of three polynomials.

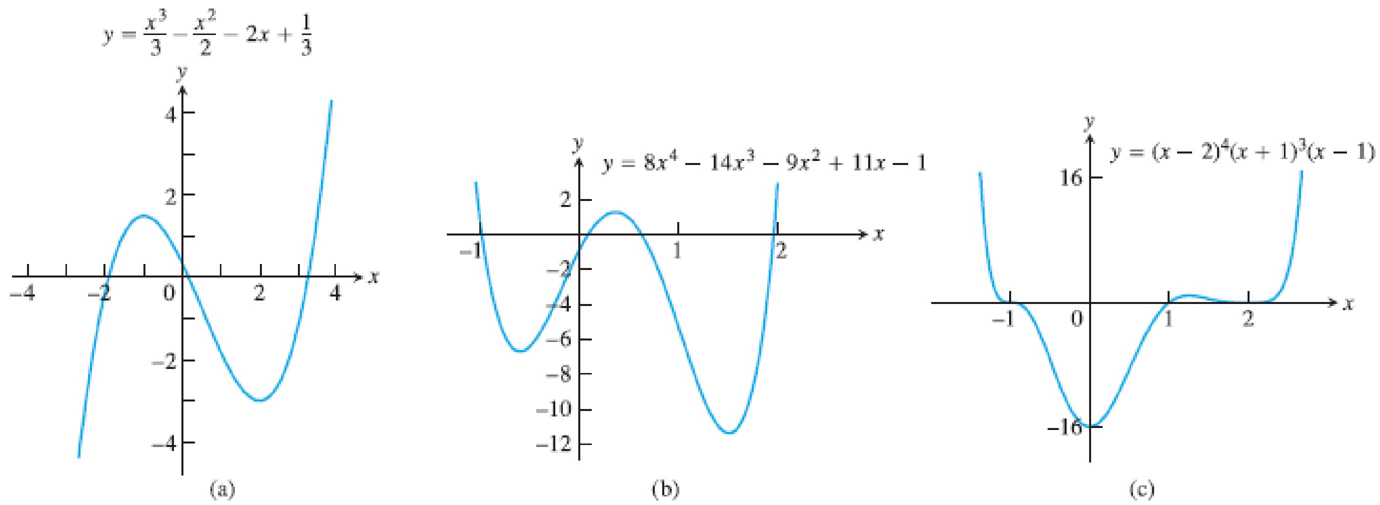


FIGURE 1.17 Graphs of three polynomial functions.

Rational Functions **A rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. The graphs of several rational functions are shown in Figure 1.18.

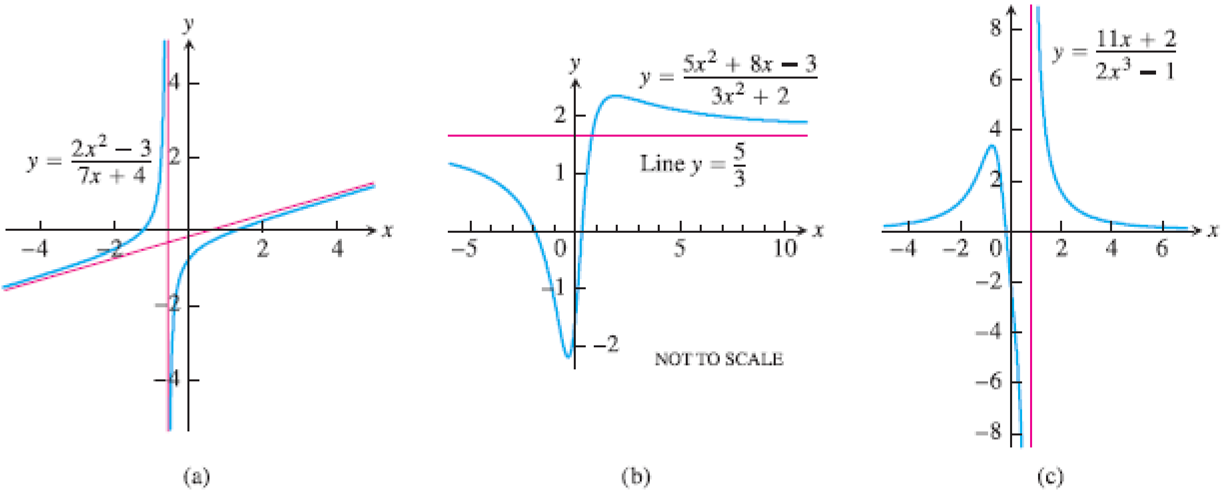


FIGURE 1.18 Graphs of three rational functions. The straight red lines are called *asymptotes* and are not part of the graph.

Algebraic Functions Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of **algebraic functions.** All rational functions are algebraic, but also included are more complicated functions (such as those satisfying an equation like $y^3 - 9xy + x^3 = 0$), Figure 1.19 displays the graphs of three algebraic functions.

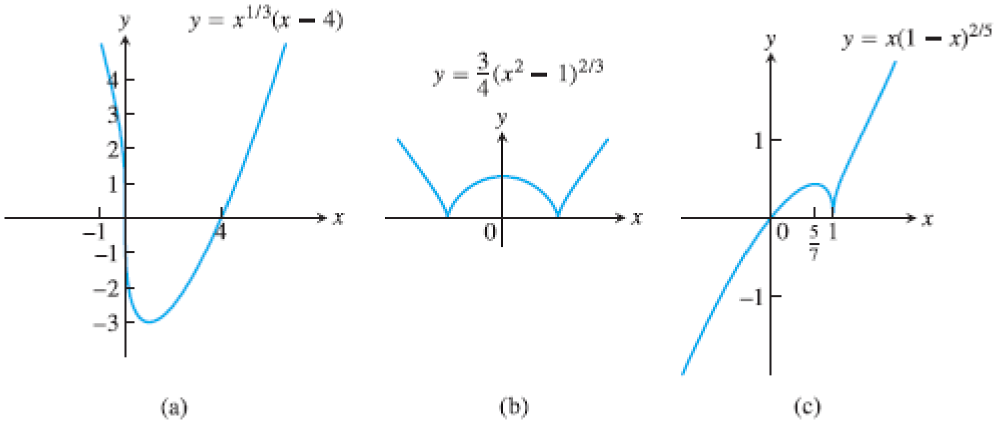


FIGURE 1.19 Graphs of three algebraic functions.

Trigonometric Functions The six basic trigonometric functions are (sine, cosine, tangent, cosecant, secant and cotangent). The graphs of the sine and cosine functions are shown in Figure 1.20.

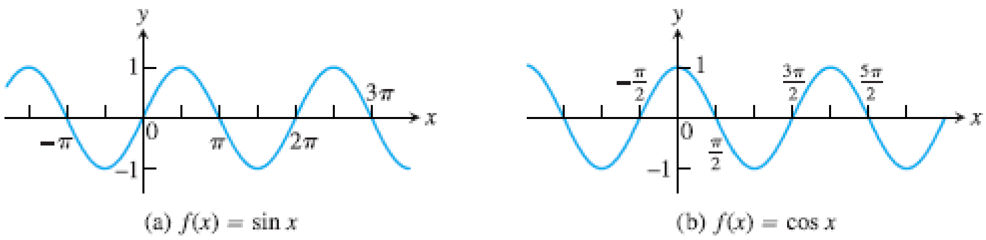


FIGURE 1.20 Graphs of the sine and cosine functions.

Exponential Functions Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and a of 1, are called **exponential functions.** All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0. The graphs of some exponential functions are shown in Figure 1.21.

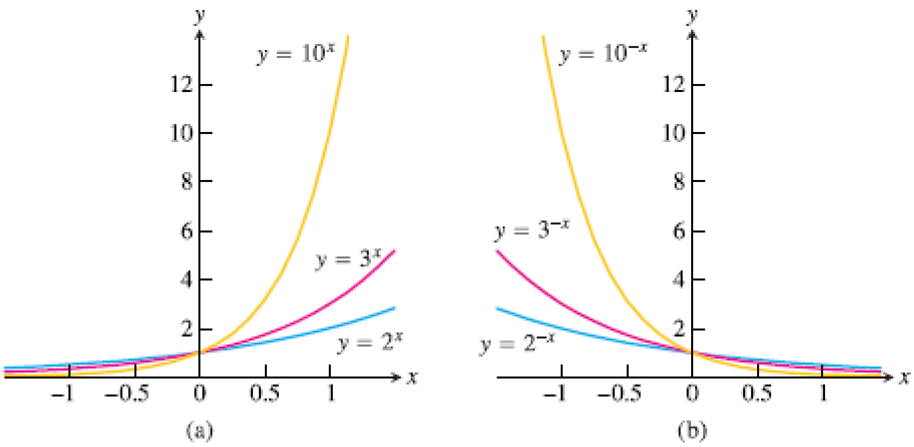


FIGURE 1.21 Graphs of exponential functions.

Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant . Figure 1.22 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

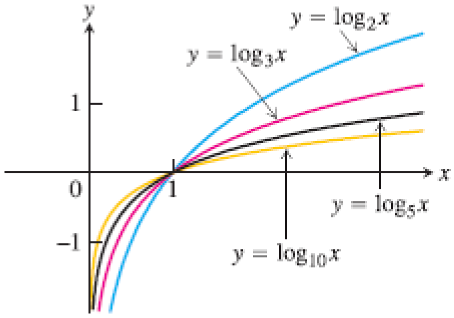


FIGURE 1.22 Graphs of four logarithmic functions.

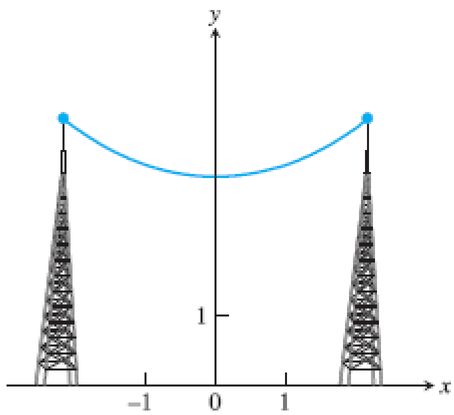


FIGURE 1.23 Graph of a catenary or hanging cable.

(The Latin word *catena* means "chain.")

Transcendental Functions These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well. A particular example of a transcendental function is a catenary.
Its graph has the shape of a cable, like a telephone line or electric cable, strung from one support to another and hanging freely under its own weight (Figure 1.23).

2.1 Rates of Change and Tangents to Curves

Calculus is a tool to help us understand how functional relationships change, such as the position or speed of a moving object as a function of time, or the changing slope of a curve being traversed by a point moving along it. In this section we introduce the ideas of average and instantaneous rates of change, and show that they are closely related to the slope of a curve at a point P on the curve. We give precise developments of these important concepts in the next chapter, but for now we use an informal approach so you will see how they lead naturally to the main idea of the chapter, the *limit*. You will see that limits play a major role in calculus and the study of change.

Average and Instantaneous Speed

HISTORICAL BIOGRAPHY*

Galileo Galilei
(1564–1642)

In the late sixteenth century, Galileo discovered that a solid object dropped from rest (not moving) near the surface of the earth and allowed to fall freely will fall a distance proportional to the square of the time it has been falling. This type of motion is called **free fall**. It assumes negligible air resistance to slow the object down, and that gravity is the only force acting on the falling body. If y denotes the distance fallen in feet after t seconds, then Galileo's law is

$$y = 16t^2,$$

where 16 is the (approximate) constant of proportionality. (If y is measured in meters, the constant is 4.9.)

A moving body's **average speed** during an interval of time is found by dividing the distance covered by the time elapsed. The unit of measure is length per unit time: kilometers per hour, feet (or meters) per second, or whatever is appropriate to the problem at hand.

EXAMPLE 1 A rock breaks loose from the top of a tall cliff. What is its average speed

(a) during the first 2 sec of fall?

(b) during the 1-sec interval between second 1 and second 2?

Solution The average speed of the rock during a given time interval is the change in distance, Δy, divided by the length of the time interval, Δt. (Increments like Δy and Δt are reviewed in Appendix 3.) Measuring distance in feet and time in seconds, we have the following calculations:

(a) For the first 2 sec: $\dfrac{\Delta y}{\Delta t} = \dfrac{16(2)^2 - 16(0)^2}{2 - 0} = 32 \dfrac{\text{ft}}{\text{sec}}$

(b) From sec 1 to sec 2: $\dfrac{\Delta y}{\Delta t} = \dfrac{16(2)^2 - 16(1)^2}{2 - 1} = 48 \dfrac{\text{ft}}{\text{sec}}$ ■

We want a way to determine the speed of a falling object at a single instant t_0, instead of using its average speed over an interval of time. To do this, we examine what happens when we calculate the average speed over shorter and shorter time intervals starting at t_0. The next example illustrates this process. Our discussion is informal here, but it will be made precise in Chapter 3.

EXAMPLE 2 Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ sec.

Solution We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$, having length $\Delta t = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16{t_0}^2}{h}. \tag{1}$$

We cannot use this formula to calculate the "instantaneous" speed at the exact moment t_0 by simply substituting $h = 0$, because we cannot divide by zero. But we *can* use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$. When we do so, we see a pattern (Table 2.1).

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

Average speed: $\dfrac{\Delta y}{\Delta t} = \dfrac{16(t_0 + h)^2 - 16{t_0}^2}{h}$

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 32 as the length of the interval decreases. This suggests that the rock is falling at a speed of 32 ft/sec at $t_0 = 1$ sec. Let's confirm this algebraically.

If we set $t_0 = 1$ and then expand the numerator in Equation (1) and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(1+2h+h^2) - 16}{h}$$

$$= \frac{32h + 16h^2}{h} = 32 + 16h.$$

For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $32 + 16h$ ft/sec. We can now see why the average speed has the limiting value $32 + 16(0) = 32$ ft/sec as h approaches 0.

Similarly, setting $t_0 = 2$ in Equation (1), the procedure yields

$$\frac{\Delta y}{\Delta t} = 64 + 16h$$

for values of h different from 0. As h gets closer and closer to 0, the average speed has the limiting value 64 ft/sec when $t_0 = 2$ sec, as suggested by Table 2.1.

The average speed of a falling object is an example of a more general idea which we discuss next.

Average Rates of Change and Secant Lines

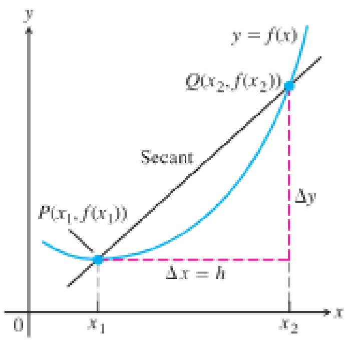


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Given an arbitrary function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y, $\Delta y = f(x_2) - f(x_1)$, by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs. (We use the symbol h for Δx to simplify the notation here and later on.)

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Figure 2.1). In geometry, a line joining two points of a curve is a **secant** to the curve. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant PQ. Let's consider what happens as the point Q approaches the point P along the curve, so the length h of the interval over which the change occurs approaches zero.

Defining the Slope of a Curve

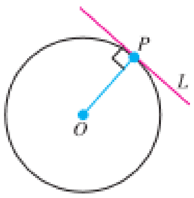


FIGURE 2.2 L is tangent to the circle at P if it passes through P perpendicular to radius OP.

We know what is meant by the slope of a straight line, which tells us the rate at which it rises or falls—its rate of change as the graph of a linear function. But what is meant by the *slope of a curve* at a point P on the curve? If there is a *tangent* line to the curve at P—a line that just touches the curve like the tangent to a circle—it would be reasonable to identify *the slope of the tangent* as the slope of the curve at P. So we need a precise meaning for the tangent at a point on a curve.

For circles, tangency is straightforward. A line L is tangent to a circle at a point P if L passes through P perpendicular to the radius at P (Figure 2.2). Such a line just *touches* the circle. But what does it mean to say that a line L is tangent to some other curve C at a point P?

To define tangency for general curves, we need an approach that takes into account the behavior of the secants through P and nearby points Q as Q moves toward P along the curve (Figure 2.3). Here is the idea:

1. Start with what we *can* calculate, namely the slope of the secant PQ.
2. Investigate the limiting value of the secant slope as Q approaches P along the curve. (We clarify the *limit* idea in the next section.)
3. If the *limit* exists, take it to be the slope of the curve at P and *define* the tangent to the curve at P to be the line through P with this slope.

This procedure is what we were doing in the falling-rock problem discussed in Example 2. The next example illustrates the geometric idea for the tangent to a curve.

HISTORICAL BIOGRAPHY

Pierre de Fermat
(1601–1665)

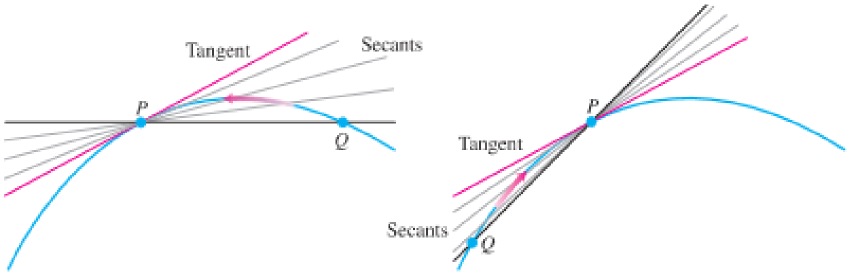


FIGURE 2.3 The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

EXAMPLE 3 Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

Solution We begin with a secant line through $P(2, 4)$ and $Q(2 + h, (2 + h)^2)$ nearby. We then write an expression for the slope of the secant PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\begin{aligned}\text{Secant slope} = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2^2}{h} &= \frac{h^2 + 4h + 4 - 4}{h} \\ &= \frac{h^2 + 4h}{h} = h + 4.\end{aligned}$$

If $h > 0$, then Q lies above and to the right of P, as in Figure 2.4. If $h < 0$, then Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant slope $h + 4$ approaches 4. We take 4 to be the parabola's slope at P.

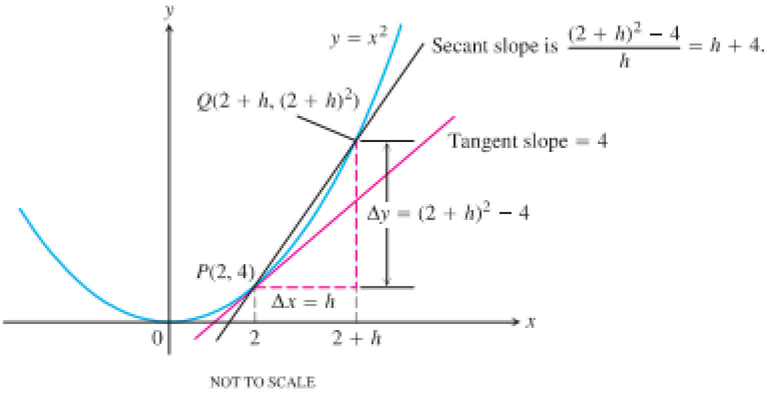


FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant slopes (Example 3).

The tangent to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2) \qquad \text{Point-slope equation}$$
$$y = 4x - 4.$$

■

Instantaneous Rates of Change and Tangent Lines

The rates at which the rock in Example 2 was falling at the instants $t = 1$ and $t = 2$ are called *instantaneous rates of change*. Instantaneous rates and slopes of tangent lines are intimately connected, as we will now see in the following examples.

EXAMPLE 4 Figure 2.5 shows how a population p of fruit flies (*Drosophila*) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time t, and the points joined by a smooth curve (colored blue in Figure 2.5). Find the average growth rate from day 23 to day 45.

Solution There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by $340 - 150 = 190$ in $45 - 23 = 22$ days. The average rate of change of the population from day 23 to day 45 was

$$\text{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day}.$$

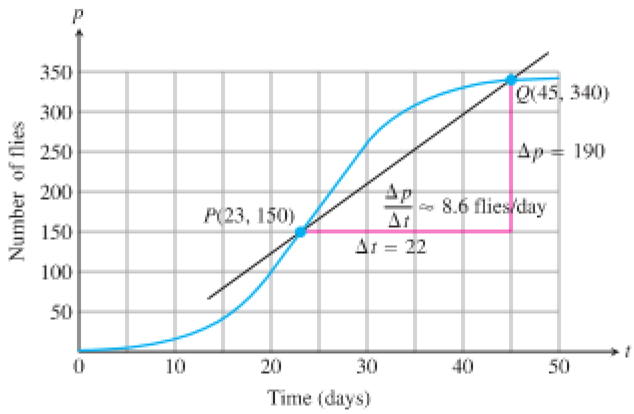


FIGURE 2.5 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line (Example 4).

This average is the slope of the secant through the points P and Q on the graph in Figure 2.5. ■

The average rate of change from day 23 to day 45 calculated in Example 4 does not tell us how fast the population was changing on day 23 itself. For that we need to examine time intervals closer to the day in question.

EXAMPLE 5 How fast was the number of flies in the population of Example 4 growing on day 23?

Solution To answer this question, we examine the average rates of change over increasingly short time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secants from P to Q, for a sequence of points Q approaching P along the curve (Figure 2.6).

Q	Slope of $PQ = \Delta p / \Delta t$ (flies/day)
(45, 340)	$\dfrac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\dfrac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\dfrac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\dfrac{265 - 150}{30 - 23} \approx 16.4$

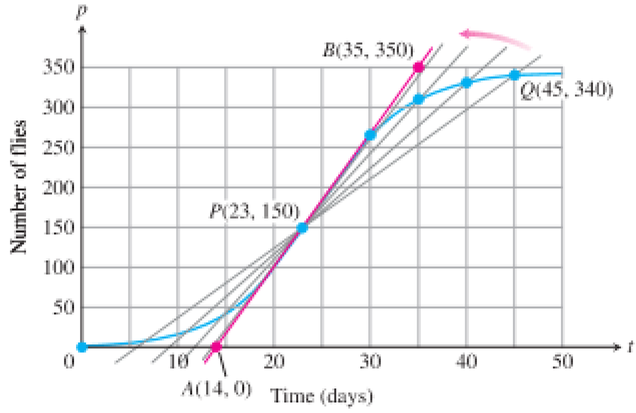


FIGURE 2.6 The positions and slopes of four secants through the point P on the fruit fly graph (Example 5).

The values in the table show that the secant slopes rise from 8.6 to 16.4 as the t-coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued on toward 23. Geometrically, the secants rotate about P and seem to approach the red tangent line in the figure. Since the line appears to pass through the points (14, 0) and (35, 350), it has slope

$$\frac{350 - 0}{35 - 14} = 16.7 \text{ flies/day (approximately).}$$

On day 23 the population was increasing at a rate of about 16.7 flies/day. ■

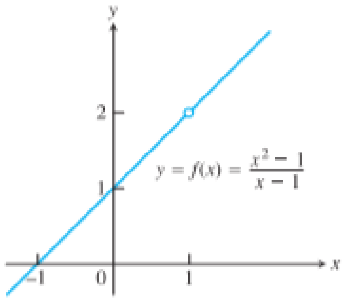
The instantaneous rates in Example 2 were found to be the values of the average speeds, or average rates of change, as the time interval of length h approached 0. That is, the instantaneous rate is the value the average rate approaches as the length h of the interval over which the change occurs approaches zero. The average rate of change corresponds to the slope of a secant line; the instantaneous rate corresponds to the slope of the tangent line as the independent variable approaches a fixed value. In Example 2, the independent variable t approached the values $t = 1$ and $t = 2$. In Example 3, the independent variable x approached the value $x = 2$. So we see that instantaneous rates and slopes of tangent lines are closely connected. We investigate this connection thoroughly in the next chapter, but to do so we need the concept of a *limit*.

2.2 Limit of a Function and Limit Laws

In Section 2.1 we saw that limits arise when finding the instantaneous rate of change of a function or the tangent to a curve. Here we begin with an informal definition of *limit* and show how we can calculate the values of limits. A precise definition is presented in the next section.

HISTORICAL ESSAY

Limits



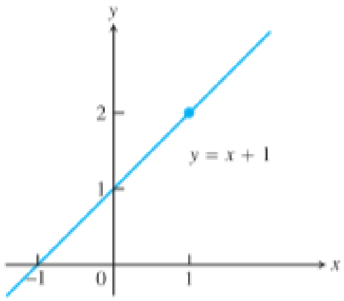


FIGURE 2.7 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

Limits of Function Values

Frequently when studying a function $y = f(x)$, we find ourselves interested in the function's behavior *near* a particular point x_0, but not *at* x_0. This might be the case, for instance, if x_0 is an irrational number, like π or $\sqrt{2}$, whose values can only be approximated by "close" rational numbers at which we actually evaluate the function instead. Another situation occurs when trying to evaluate a function at x_0 leads to division by zero, which is undefined. We encountered this last circumstance when seeking the instantaneous rate of change in y by considering the quotient function $\Delta y/h$ for h closer and closer to zero. Here's a specific example where we explore numerically how a function behaves near a particular point at which we cannot directly evaluate the function.

EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \qquad \text{for} \qquad x \neq 1.$$

The graph of f is the line $y = x + 1$ with the point $(1, 2)$ *removed*. This removed point is shown as a "hole" in Figure 2.7. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ *as close as we want* to 2 by choosing x *close enough* to 1 (Table 2.2). ■

TABLE 2.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \dfrac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

Let's generalize the idea illustrated in Example 1.

Suppose $f(x)$ is defined on an open interval about x_0, *except possibly at x_0 itself*. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches x_0 is L." For instance, in Example 1 we would say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Essentially, the definition says that the values of $f(x)$ are close to the number L whenever x is close to x_0 (on either side of x_0). This definition is "informal" because phrases like *arbitrarily close* and *sufficiently close* are imprecise; their meaning depends on the context. (To a machinist manufacturing a piston, *close* may mean *within a few thousandths of an inch*. To an astronomer studying distant galaxies, *close* may mean *within a few thousand light-years*.) Nevertheless, the definition is clear enough to enable us to recognize and evaluate limits of specific functions. We will need the precise definition of Section 2.3, however, when we set out to prove theorems about limits. Here are several more examples exploring the idea of limits.

EXAMPLE 2 This example illustrates that the limit value of a function does not depend on how the function is defined at the point being approached. Consider the three functions in Figure 2.8. The function f has limit 2 as $x \to 1$ even though f is not defined at $x = 1$.

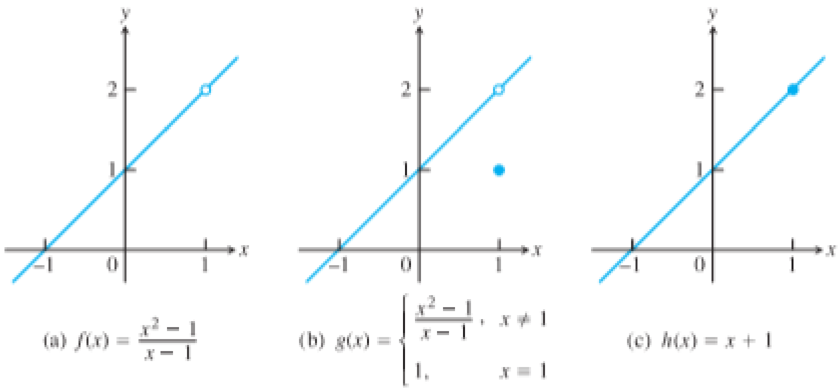
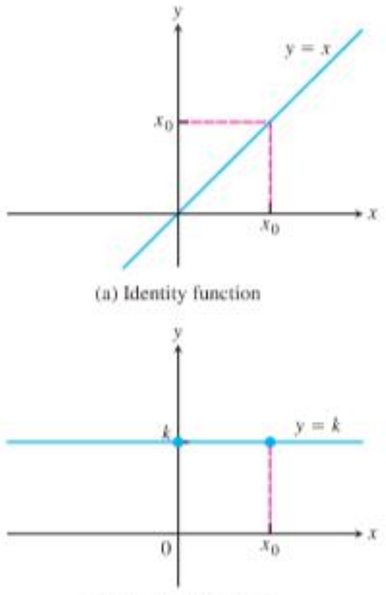


FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).



(a) Identity function

(b) Constant function

FIGURE 2.9 The functions in Example 3 have limits at all points x_0.

The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$. The function h is the only one of the three functions in Figure 2.8 whose limit as $x \to 1$ equals its value at $x = 1$. For h, we have $\lim_{x \to 1} h(x) = h(1)$. This equality of limit and function value is significant, and we return to it in Section 2.5. ■

EXAMPLE 3

(a) If f is the **identity function** $f(x) = x$, then for any value of x_0 (Figure 2.9a),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0.$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0 (Figure 2.9b),

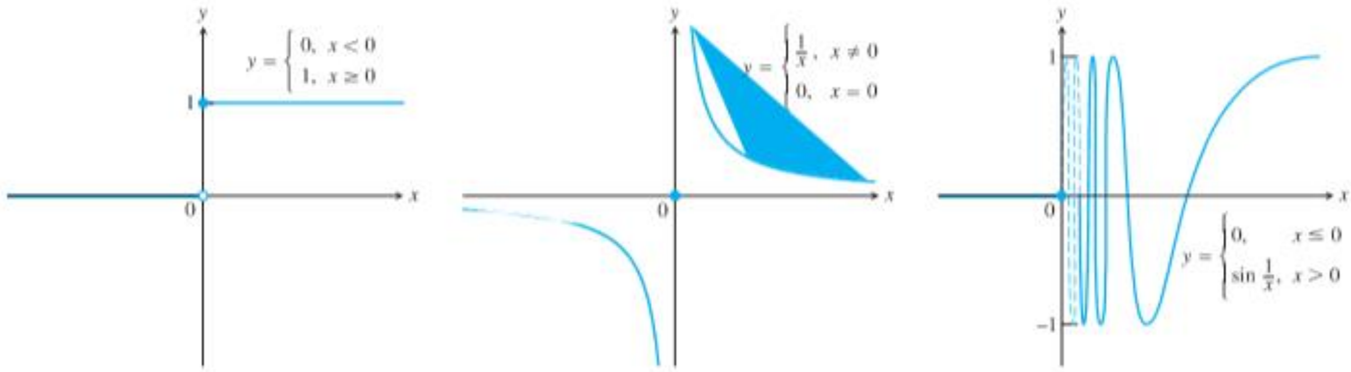
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k.$$

For instances of each of these rules we have

$$\lim_{x \to 3} x = 3 \qquad \text{and} \qquad \lim_{x \to -7} (4) = \lim_{x \to 2} (4) = 4.$$

We prove these rules in Example 3 in Section 2.3. ■

Some ways that limits can fail to exist are illustrated in Figure 2.10 and described in the next example.



(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

EXAMPLE 4 Discuss the behavior of the following functions as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \dfrac{1}{x}, & x > 0 \end{cases}$

Solution

(a) It *jumps*: The **unit step function** $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \to 0$ (Figure 2.10a).

(b) It *grows too "large" to have a limit*: $g(x)$ has no limit as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and do not stay close to *any* fixed real number (Figure 2.10b).

(c) It *oscillates too much to have a limit*: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$ (Figure 2.10c). ■

The Limit Laws

When discussing limits, sometimes we use the notation $x \to x_0$ if we want to emphasize the point x_0 that is being approached in the limit process (usually to enhance the clarity of a particular discussion or example). Other times, such as in the statements of the following theorem, we use the simpler notation $x \to c$ or $x \to a$ which avoids the subscript in x_0. In every case, the symbols x_0, c, and a refer to a single point on the x-axis that may or may not belong to the domain of the function involved. To calculate limits of functions that are arithmetic combinations of functions having known limits, we can use several easy rules.

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

EXAMPLE 5 Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$ (Example 3) and the properties of limits to find the following limits.

(a) $\lim_{x \to c}(x^3 + 4x^2 - 3)$ **(b)** $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$ **(c)** $\lim_{x \to -2} \sqrt{4x^2 - 3}$

Solution

(a)
$$\lim_{x \to c}(x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3 \qquad \text{Sum and Difference Rules}$$
$$= c^3 + 4c^2 - 3 \qquad \text{Power and Multiple Rules}$$

(b)
$$\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c}(x^4 + x^2 - 1)}{\lim_{x \to c}(x^2 + 5)} \qquad \text{Quotient Rule}$$
$$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5} \qquad \text{Sum and Difference Rules}$$
$$= \frac{c^4 + c^2 - 1}{c^2 + 5} \qquad \text{Power or Product Rule}$$

(c)
$$\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2}(4x^2 - 3)} \qquad \text{Root Rule with } n = 2$$
$$= \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3} \qquad \text{Difference Rule}$$
$$= \sqrt{4(-2)^2 - 3} \qquad \text{Product and Multiple Rules}$$
$$= \sqrt{16 - 3}$$
$$= \sqrt{13}$$

Two consequences of Theorem 1 further simplify the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial function as x approaches c, merely substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point c *at which the denominator is not zero*, substitute c for x in the formula for the function. (See Examples 5a and 5b.) We state these results formally as theorems.

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 6

The following calculation illustrates Theorems 2 and 3:

$$\lim_{x\to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Identifying Common Factors
It can be shown that if $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.

Eliminating Zero Denominators Algebraically

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at c. If this happens, we can find the limit by substitution in the simplified fraction.

EXAMPLE 7

Evaluate

$$\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling the $(x - 1)$'s gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by substitution:

$$\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x\to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

See Figure 2.11.

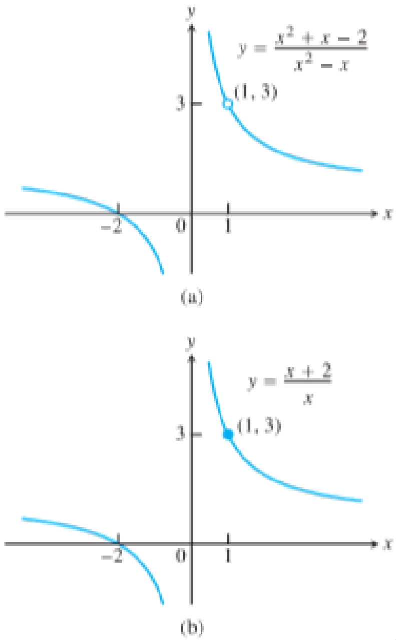


FIGURE 2.11 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 7).

Using Calculators and Computers to Estimate Limits

When we cannot use the Quotient Rule in Theorem 1 because the limit of the denominator is zero, we can try using a calculator or computer to guess the limit numerically as x gets closer and closer to c. We used this approach in Example 1, but calculators and computers can sometimes give false values and misleading impressions for functions that are undefined at a point or fail to have a limit there, as we now illustrate.

EXAMPLE 8

Estimate the value of $\lim_{x\to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$.

Solution Table 2.3 lists values of the function for several values near $x = 0$. As x approaches 0 through the values ± 1, ± 0.5, ± 0.10, and ± 0.01, the function seems to approach the number 0.05.

As we take even smaller values of x, ± 0.0005, ± 0.0001, ± 0.00001, and ± 0.000001, the function appears to approach the value 0.

Is the answer 0.05 or 0, or some other value? We resolve this question in the next example.

TABLE 2.3 Computer values of $f(x) = \dfrac{\sqrt{x^2+100}-10}{x^2}$ near $x = 0$

x	$f(x)$	
± 1	0.049876	approaches 0.05?
± 0.5	0.049969	
± 0.1	0.049999	
± 0.01	0.050000	
± 0.0005	0.080000	approaches 0?
± 0.0001	0.000000	
± 0.00001	0.000000	
± 0.000001	0.000000	

Using a computer or calculator may give ambiguous results, as in the last example. We cannot substitute $x = 0$ in the problem, and the numerator and denominator have no obvious common factors (as they did in Example 7). Sometimes, however, we can create a common factor algebraically.

EXAMPLE 9 Evaluate

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}.$$

Solution This is the limit we considered in Example 8. We can create a common factor by multiplying both numerator and denominator by the conjugate radical expression $\sqrt{x^2+100}+10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\begin{aligned}
\frac{\sqrt{x^2+100}-10}{x^2} &= \frac{\sqrt{x^2+100}-10}{x^2} \cdot \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10} \\
&= \frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)} \\
&= \frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)} && \text{Common factor } x^2 \\
&= \frac{1}{\sqrt{x^2+100}+10}. && \text{Cancel } x^2 \text{ for } x \neq 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} &= \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} \\
&= \frac{1}{\sqrt{0^2+100}+10} && \text{Denominator not 0 at } x = 0\text{; substitute} \\
&= \frac{1}{20} = 0.05.
\end{aligned}$$

This calculation provides the correct answer, in contrast to the ambiguous computer results in Example 8. ■

We cannot always algebraically resolve the problem of finding the limit of a quotient where the denominator becomes zero. In some cases the limit might then be found with the

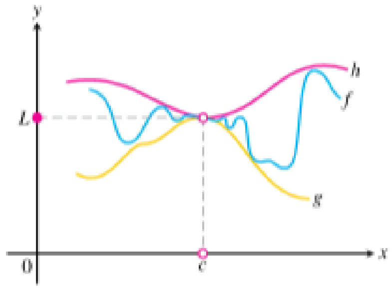


FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h.

aid of some geometry applied to the problem (see the proof of Theorem 7 in Section 2.4), or through methods of calculus (illustrated in Section 7.5). The next theorem is also useful.

The Sandwich Theorem

The following theorem enables us to calculate a variety of limits. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c. Being trapped between the values of two functions that approach L, the values of f must also approach L (Figure 2.12). You will find a proof in Appendix 4.

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem.

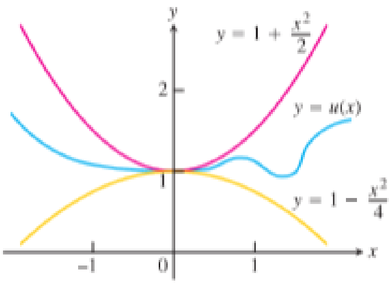


FIGURE 2.13 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$ (Example 10).

EXAMPLE 10 Given that

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \qquad \text{for all } x \ne 0,$$

find $\lim_{x \to 0} u(x)$, no matter how complicated u is.

Solution Since

$$\lim_{x \to 0} (1 - (x^2/4)) = 1 \qquad \text{and} \qquad \lim_{x \to 0} (1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x \to 0} u(x) = 1$ (Figure 2.13). ■

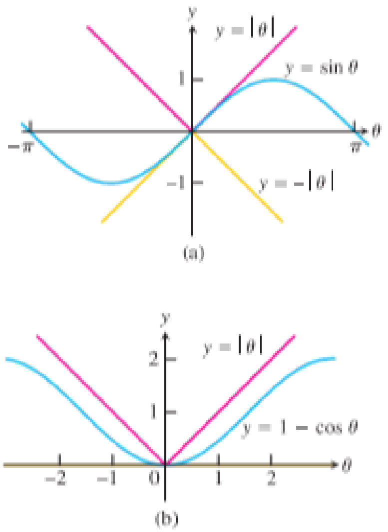


FIGURE 2.14 The Sandwich Theorem confirms the limits in Example 11.

EXAMPLE 11 The Sandwich Theorem helps us establish several important limit rules:

(a) $\lim_{\theta \to 0} \sin\theta = 0$ **(b)** $\lim_{\theta \to 0} \cos\theta = 1$

(c) For any function f, $\lim_{x \to c} |f(x)| = 0$ implies $\lim_{x \to c} f(x) = 0$.

Solution

(a) In Section 1.3 we established that $-|\theta| \le \sin\theta \le |\theta|$ for all θ (see Figure 2.14a). Since $\lim_{\theta \to 0} (-|\theta|) = \lim_{\theta \to 0} |\theta| = 0$, we have

$$\lim_{\theta \to 0} \sin\theta = 0.$$

(b) From Section 1.3, $0 \le 1 - \cos\theta \le |\theta|$ for all θ (see Figure 2.14b), and we have $\lim_{\theta \to 0} (1 - \cos\theta) = 0$ or

$$\lim_{\theta \to 0} \cos\theta = 1.$$

(c) Since $-|f(x)| \le f(x) \le |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \to c$, it follows that $\lim_{x \to c} f(x) = 0$. ■

Another important property of limits is given by the next theorem. A proof is given in the next section.

THEOREM 5 If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x).$$

The assertion resulting from replacing the less than or equal to ($\le$) inequality by the strict less than ($<$) inequality in Theorem 5 is false. Figure 2.14a shows that for $\theta \neq 0$, $-|\theta| < \sin\theta < |\theta|$, but in the limit as $\theta \to 0$, equality holds.

DIFFERENTIATION

3.1 Tangents and the Derivative at a Point

In this section we define the slope and tangent to a curve at a point, and the derivative of a function at a point. Later in the chapter we interpret the derivative as the instantaneous rate of change of a function, and apply this interpretation to the study of certain types of motion.

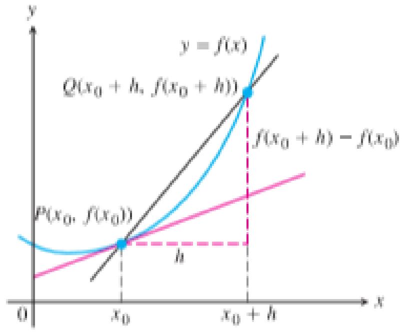


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h\to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$.

Finding a Tangent to the Graph of a Function

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the procedure introduced in Section 2.1. We calculate the slope of the secant through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 3.1). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

In Section 2.1, Example 3, we applied these definitions to find the slope of the parabola $f(x) = x^2$ at the point $P(2, 4)$ and the tangent line to the parabola at P. Let's look at another example.

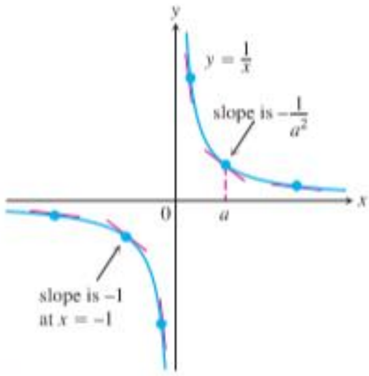


FIGURE 3.2 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away (Example 1).

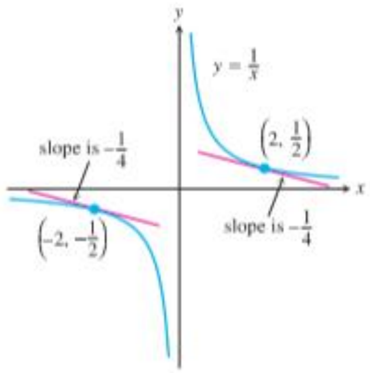


FIGURE 3.3 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 1).

EXAMPLE 1

(a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution

(a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h} = \lim_{h \to 0} \frac{1}{h} \frac{a - (a+h)}{a(a+h)}$$

$$= \lim_{h \to 0} \frac{-h}{ha(a+h)} = \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim_{h \to 0}$" before each fraction until the stage where we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0. When $a = -1$, the slope is $-1/(-1)^2 = -1$ (Figure 3.2).

(b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided that

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 3.3).

(c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Figure 3.2). We see this situation again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent levels off to become horizontal. ■

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}, \quad h \neq 0$$

is called the **difference quotient of f at x_0 with increment h**. If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

If we interpret the difference quotient as the slope of a secant line, then the derivative gives the slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$. Exercise 31 shows that the derivative of the linear function $f(x) = mx + b$ at any point x_0 is simply the slope of the line, so

$$f'(x_0) = m,$$

which is consistent with our definition of slope.

If we interpret the difference quotient as an average rate of change (Section 2.1), the derivative gives the function's instantaneous rate of change with respect to x at the point $x = x_0$. We study this interpretation in Section 3.4.

EXAMPLE 2 In Examples 1 and 2 in Section 2.1, we studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell $y = 16t^2$ feet during the first t sec, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. What was the rock's *exact* speed at this time?

Solution We let $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds, for $h > 0$, was found to be

$$\frac{f(1 + h) - f(1)}{h} = \frac{16(1 + h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h + 2).$$

The rock's speed at the instant $t = 1$ is then

$$\lim_{h \to 0} 16(h + 2) = 16(0 + 2) = 32 \text{ ft/sec.}$$

Our original estimate of 32 ft/sec in Section 2.1 was right. ■

Summary

We have been discussing slopes of curves, lines tangent to a curve, the rate of change of a function, and the derivative of a function at a point. All of these ideas refer to the same limit.

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

In the next sections, we allow the point x_0 to vary across the domain of the function f.

3.2 The Derivative as a Function

HISTORICAL ESSAY

The Derivative

In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f.

> **DEFINITION** The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$
>
> provided the limit exists.

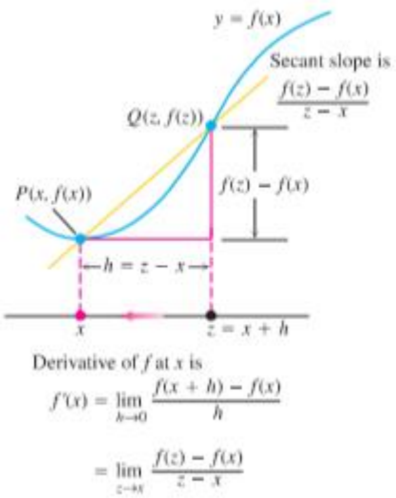


FIGURE 3.4 Two forms for the difference quotient.

We use the notation $f(x)$ in the definition to emphasize the independent variable x with respect to which the derivative function $f'(x)$ is being defined. The domain of f' is the set of points in the domain of f for which the limit exists, which means that the domain may be the same as or smaller than the domain of f. If f' exists at a particular x, we say that f is **differentiable (has a derivative) at x**. If f' exists at every point in the domain of f, we call f **differentiable**.

If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x. Therefore, an equivalent definition of the derivative is as follows (see Figure 3.4). This formula is sometimes more convenient to use when finding a derivative function.

> **Alternative Formula for the Derivative**
>
> $$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

as another way to denote the derivative $f'(x)$. Example 1 of Section 3.1 illustrated the differentiation process for the function $y = 1/x$ when $x = a$. For x representing any point in the domain, we get the formula

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

> **Derivative of the Reciprocal Function**
>
> $$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad x \neq 0$$

Here are two more examples in which we allow x to be any point in the domain of f.

EXAMPLE 1 Differentiate $f(x) = \dfrac{x}{x - 1}$.

Solution We use the definition of derivative, which requires us to calculate $f(x + h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f(x) = \frac{x}{x - 1} \quad \text{and} \quad f(x + h) = \frac{(x + h)}{(x + h) - 1}, \text{ so}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \qquad \text{Definition}$$

$$= \lim_{h \to 0} \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} \qquad \text{Simplify}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. \qquad \text{Cancel } h \neq 0$$

EXAMPLE 2

(a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Derivative of the Square Root Function

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0$$

Solution

(a) We use the alternative formula to calculate f':

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

$$= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x}$$

$$= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{\left(\sqrt{z} - \sqrt{x}\right)\left(\sqrt{z} + \sqrt{x}\right)}$$

$$= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

(b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line through the point (4, 2) with slope 1/4 (Figure 3.5):

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1.$$

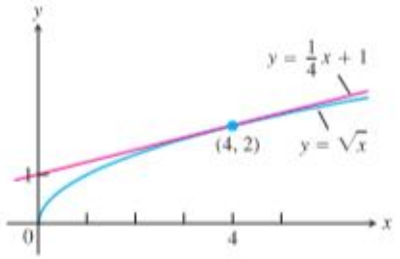


FIGURE 3.5 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

The symbols d/dx and D indicate the operation of differentiation. We read dy/dx as "the derivative of y with respect to x," and df/dx and $(d/dx)f(x)$ as "the derivative of f with respect to x." The "prime" notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. The symbol dy/dx should not be regarded as a ratio

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

THEOREM 1—Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof Given that $f'(c)$ exists, we must show that $\lim_{x \to c} f(x) = f(c)$, or equivalently, that $\lim_{h \to 0} f(c + h) = f(c)$. If $h \neq 0$, then

$$\begin{aligned} f(c + h) &= f(c) + (f(c + h) - f(c)) \\ &= f(c) + \frac{f(c + h) - f(c)}{h} \cdot h . \end{aligned}$$

Now take limits as $h \to 0$. By Theorem 1 of Section 2.2,

$$\begin{aligned} \lim_{h \to 0} f(c + h) &= \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \cdot \lim_{h \to 0} h \\ &= f(c) + f'(c) \cdot 0 \\ &= f(c) + 0 \\ &= f(c). \end{aligned}$$

3.3 Differentiation Rules

This section introduces several rules that allow us to differentiate constant functions, power functions, polynomials, rational functions, and certain combinations of them, simply and directly, without having to take limits each time.

Powers, Multiples, Sums, and Differences

A simple rule of differentiation is that the derivative of every constant function is zero.

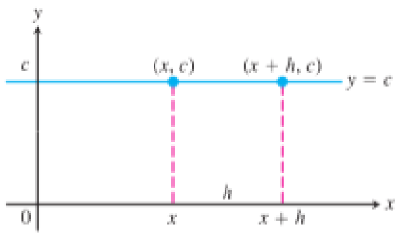


FIGURE 3.9 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

Derivative of a Constant Function
If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Proof We apply the definition of the derivative to $f(x) = c$, the function whose outputs have the constant value c (Figure 3.9). At every value of x, we find that

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0. \qquad \blacksquare$$

From Section 3.1, we know that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad \text{or} \quad \frac{d}{dx}\left(x^{-1}\right) = -x^{-2}.$$

From Section 3.1, we know that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad \text{or} \quad \frac{d}{dx}\left(x^{-1}\right) = -x^{-2}.$$

From Example 2 of the last section we also know that

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}, \quad \text{or} \quad \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

These two examples illustrate a general rule for differentiating a power x^n. We first prove the rule when n is a positive integer.

Power Rule for Positive Integers:
If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

HISTORICAL BIOGRAPHY

Richard Courant
(1888–1972)

Proof of the Positive Integer Power Rule The formula

$$z^n - x^n = (z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})$$

can be verified by multiplying out the right-hand side. Then from the alternative formula for the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{z^n - x^n}{z - x} \\ &= \lim_{z \to x}(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}) \qquad n \text{ terms} \\ &= nx^{n-1}. \end{aligned}$$

■

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

EXAMPLE 1 Differentiate the following powers of x.

(a) x^3 **(b)** $x^{2/3}$ **(c)** $x^{\sqrt{2}}$ **(d)** $\frac{1}{x^4}$ **(e)** $x^{-4/3}$ **(f)** $\sqrt{x^{2+\pi}}$

Solution

(a) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$ **(b)** $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$

(c) $\frac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$ **(d)** $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$

(e) $\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3}$

(f) $\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$ ■

The next rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

In particular, if n is any real number, then

$$\frac{d}{dx}(cx^n) = cnx^{n-1}.$$

Proof

$$\frac{d}{dx}cu = \lim_{h\to 0}\frac{cu(x+h) - cu(x)}{h} \qquad \text{Derivative definition with } f(x) = cu(x)$$

$$= c\lim_{h\to 0}\frac{u(x+h) - u(x)}{h} \qquad \text{Constant Multiple Limit Property}$$

$$= c\frac{du}{dx} \qquad u \text{ is differentiable.}$$ ■

EXAMPLE 2

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.10).

(b) Negative of a function

The derivative of the negative of a differentiable function u is the negative of the function's derivative. The Constant Multiple Rule with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives.

Denoting Functions by u and v
The functions we are working with when we need a differentiation formula are likely to be denoted by letters like f and g. We do not want to use these same letters when stating general differentiation rules, so we use letters like u and v instead that are not likely to be already in use.

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

For example, if $y = x^4 + 12x$, then y is the sum of $u(x) = x^4$ and $v(x) = 12x$. We then have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) = 4x^3 + 12.$$

Proof We apply the definition of the derivative to $f(x) = u(x) + v(x)$:

$$\begin{aligned}
\frac{d}{dx}[u(x) + v(x)] &= \lim_{h \to 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\
&= \lim_{h \to 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right] \\
&= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}.
\end{aligned}$$

Combining the Sum Rule with the Constant Multiple Rule gives the **Difference Rule**, which says that the derivative of a *difference* of differentiable functions is the difference of their derivatives:

$$\frac{d}{dx}(u - v) = \frac{d}{dx}[u + (-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}.$$

The Sum Rule also extends to finite sums of more than two functions. If $u_1, u_2, \ldots, u_n$ are differentiable at x, then so is $u_1 + u_2 + \cdots + u_n$, and

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.$$

For instance, to see that the rule holds for three functions we compute

$$\frac{d}{dx}(u_1 + u_2 + u_3) = \frac{d}{dx}((u_1 + u_2) + u_3) = \frac{d}{dx}(u_1 + u_2) + \frac{du_3}{dx} = \frac{du_1}{dx} + \frac{du_2}{dx} + \frac{du_3}{dx}.$$

EXAMPLE 3 Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

Solution
$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \qquad \text{Sum and Difference Rules}$$
$$= 3x^2 + \frac{4}{3}\cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$

EXAMPLE 4 Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution The horizontal tangents, if any, occur where the slope dy/dx is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

Now solve the equation $\frac{dy}{dx} = 0$ for x:

$$4x^3 - 4x = 0$$
$$4x(x^2 - 1) = 0$$
$$x = 0, 1, -1.$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1$, and -1. The corresponding points on the curve are $(0, 2)$, $(1, 1)$ and $(-1, 1)$. See Figure 3.11. We will see in Chapter 4 that finding the values of x where the derivative of a function is equal to zero is an important and useful procedure. ■

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \qquad \text{while} \qquad \frac{d}{dx}(x)\cdot\frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product of two functions is the sum of *two* products, as we now explain.

Derivative Product Rule
If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The derivative of the product uv is u times the derivative of v plus v times the derivative of u. In *prime notation*, $(uv)' = uv' + vu'$. In function notation,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

EXAMPLE 5 Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution

(a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}\left[(x^2 + 1)(x^3 + 3)\right] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x) \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= 3x^4 + 3x^2 + 2x^4 + 6x$$
$$= 5x^4 + 3x^2 + 6x.$$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$$
$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

This is in agreement with our first calculation. ■

Proof of the Derivative Product Rule

$$\frac{d}{dx}(uv) = \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add $u(x+h)v(x)$ in the numerator:

$$\begin{aligned}
\frac{d}{dx}(uv) &= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} \left[u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h} \right] \\
&= \lim_{h\to 0} u(x+h) \cdot \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} + v(x) \cdot \lim_{h\to 0} \frac{u(x+h) - u(x)}{h}.
\end{aligned}$$

As h approaches zero, $u(x+h)$ approaches $u(x)$ because u, being differentiable at x, is continuous at x. The two fractions approach the values of dv/dx at x and du/dx at x. In short,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$ ■

The derivative of the quotient of two functions is given by the Quotient Rule.

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

In function notation,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

EXAMPLE 6 Find the derivative of $y = \dfrac{t^2 - 1}{t^3 + 1}$.

Solution We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{(t^3 + 1)\cdot 2t - (t^2 - 1)\cdot 3t^2}{(t^3 + 1)^2} \\
&= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2} \\
&= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}
\end{aligned}$$

$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

■

Proof of the Derivative Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h\to 0}\frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h}$$

$$= \lim_{h\to 0}\frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v, we subtract and add $v(x)u(x)$ in the numerator. We then get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h\to 0}\frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)}$$

$$= \lim_{h\to 0}\frac{v(x)\dfrac{u(x+h)-u(x)}{h} - u(x)\dfrac{v(x+h)-v(x)}{h}}{v(x+h)v(x)}.$$

Taking the limits in the numerator and denominator now gives the Quotient Rule. ■

The choice of which rules to use in solving a differentiation problem can make a difference in how much work you have to do. Here is an example.

EXAMPLE 7 Rather than using the Quotient Rule to find the derivative of

$$y = \frac{(x-1)(x^2-2x)}{x^4},$$

expand the numerator and divide by x^4:

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4}$$

$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$

■

INTEGRATION

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The Fundamental Theorem of Calculus

THEOREM 3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

EXAMPLE 1 Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

Solution The average value of f on $[a, b]$ is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\cdot 0 = 0.$$

By the Mean Value Theorem, f assumes this value at some point $c \in [a, b]$.

THEOREM 4—The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

Before proving Theorem 4, we look at several examples to gain a better understanding of what it says. In each example, notice that the independent variable appears in a limit of integration, possibly in a formula.

EXAMPLE 2 Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1)\,dt$ **(b)** $y = \int_x^5 3t \sin t\,dt$ **(c)** $y = \int_1^{x^2} \cos t\,dt$

Solution We calculate the derivatives with respect to the independent variable x.

(a) $\dfrac{dy}{dx} = \dfrac{d}{dx}\displaystyle\int_a^x (t^3 + 1)\,dt = x^3 + 1$ Eq. (2) with $f(t) = t^3 + 1$

(b) $\dfrac{dy}{dx} = \dfrac{d}{dx}\displaystyle\int_x^5 3t\sin t\,dt = \dfrac{d}{dx}\left(-\int_5^x 3t\sin t\,dt\right)$ Table 5.4, Rule 1

$$= -\frac{d}{dx}\int_5^x 3t\sin t\,dt$$

$= -3x\sin x$ Eq. (2) with $f(t) = 3t\sin t$

(c) The upper limit of integration is not x but x^2. This makes y a composite of the two functions,

$$y = \int_1^u \cos t\,dt \quad \text{and} \quad u = x^2.$$

We must therefore apply the Chain Rule when finding dy/dx.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\
&= \left(\frac{d}{du}\int_1^u \cos t\,dt\right)\cdot\frac{du}{dx} \\
&= \cos u\cdot\frac{du}{dx} \\
&= \cos(x^2)\cdot 2x \\
&= 2x\cos x^2
\end{aligned}$$

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2 If f is continuous at every point in $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

EXAMPLE 3 We calculate several definite integrals using the Evaluation Theorem, rather than by taking limits of Riemann sums.

(a) $$\int_0^{\pi} \cos x \, dx = \sin x \Big]_0^{\pi} = \sin \pi - \sin 0 = 0 - 0 = 0$$

$\frac{d}{dx} \sin x = \cos x$

(b) $$\int_{-\pi/4}^{0} \sec x \tan x \, dx = \sec x \Big]_{-\pi/4}^{0} = \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$$

$\frac{d}{dx} \sec x = \sec x \tan x$

(c) $$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4 = \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right] = [8 + 1] - [5] = 4.$$

$\frac{d}{dx}\left(x^{3/2} + \frac{4}{x}\right) = \frac{3}{2}x^{1/2} - \frac{4}{x^2}$

■

Exercise 66 offers another proof of the Evaluation Theorem, bringing together the ideas of Riemann sums, the Mean Value Theorem, and the definition of the definite integral.

THEOREM 5—The Net Change Theorem The net change in a function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\, dx. \tag{6}$$

EXAMPLE 4 Here are several interpretations of the Net Change Theorem.

(a) If $c(x)$ is the cost of producing x units of a certain commodity, then $c'(x)$ is the marginal cost (Section 3.4). From Theorem 5,

$$\int_{x_1}^{x_2} c'(x)\, dx = c(x_2) - c(x_1),$$

which is the cost of increasing production from x_1 units to x_2 units.

(b) If an object with position function $s(t)$ moves along a coordinate line, its velocity is $v(t) = s'(t)$. Theorem 5 says that

$$\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1),$$

so the integral of velocity is the **displacement** over the time interval $t_1 \le t \le t_2$. On the other hand, the integral of the speed $|v(t)|$ is the **total distance traveled** over the time interval. This is consistent with our discussion in Section 5.1. ■

If we rearrange Equation (6) as

$$F(b) = F(a) + \int_a^b F'(x)\, dx,$$

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

EXAMPLE 8 Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x, -1 \leq x \leq 2$.

Solution First find the zeros of f. Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x + 1)(x - 2),$$

the zeros are $x = 0, -1$, and 2 (Figure 5.22). The zeros subdivide $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f \geq 0$, and $[0, 2]$, on which $f \leq 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

The total enclosed area is obtained by adding the absolute values of the calculated integrals.

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$

Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

From another point of view, this same equation says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int u^n \frac{du}{dx}\, dx = \frac{u^{n+1}}{n+1} + C. \tag{1}$$

The integral in Equation (1) is equal to the simpler integral

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C,$$

which suggests that the simpler expression du can be substituted for $(du/dx)\, dx$ when computing an integral. Leibniz, one of the founders of calculus, had the insight that indeed this substitution could be done, leading to the *substitution method* for computing integrals. As with differentials, when computing integrals we have

$$du = \frac{du}{dx}\, dx.$$

EXAMPLE 1 Find the integral $\displaystyle\int (x^3 + x)^5(3x^2 + 1)\, dx$.

Solution We set $u = x^3 + x$. Then

$$du = \frac{du}{dx}\, dx = (3x^2 + 1)\, dx,$$

so that by substitution we have

$$\begin{aligned}
\int (x^3 + x)^5(3x^2 + 1)\, dx &= \int u^5\, du && \text{Let } u = x^3 + x,\ du = (3x^2 + 1)\, dx. \\
&= \frac{u^6}{6} + C && \text{Integrate with respect to } u. \\
&= \frac{(x^3 + x)^6}{6} + C && \text{Substitute } x^3 + x \text{ for } u.
\end{aligned}$$

■

EXAMPLE 2 Find $\displaystyle\int \sqrt{2x + 1}\, dx$.

Solution The integral does not fit the formula

$$\int u^n\, du,$$

with $u = 2x + 1$ and $n = 1/2$, because

$$du = \frac{du}{dx}\,dx = 2\,dx$$

is not precisely dx. The constant factor 2 is missing from the integral. However, we can introduce this factor after the integral sign if we compensate for it by a factor of 1/2 in front of the integral sign. So we write

$$\begin{aligned}
\int \sqrt{2x+1}\,dx &= \frac{1}{2}\int \underbrace{\sqrt{2x+1}}_{u} \cdot \underbrace{2\,dx}_{du} \\
&= \frac{1}{2}\int u^{1/2}\,du && \text{Let } u = 2x+1,\ du = 2\,dx. \\
&= \frac{1}{2}\frac{u^{3/2}}{3/2} + C && \text{Integrate with respect to } u. \\
&= \frac{1}{3}(2x+1)^{3/2} + C && \text{Substitute } 2x+1 \text{ for } u.
\end{aligned}$$

The substitutions in Examples 1 and 2 are instances of the following general rule.

THEOREM 6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof By the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ whenever F is an antiderivative of f:

$$\begin{aligned}
\frac{d}{dx}F(g(x)) &= F'(g(x)) \cdot g'(x) && \text{Chain Rule} \\
&= f(g(x)) \cdot g'(x). && F' = f
\end{aligned}$$

If we make the substitution $u = g(x)$, then

$$\begin{aligned}
\int f(g(x))g'(x)\,dx &= \int \frac{d}{dx}F(g(x))\,dx \\
&= F(g(x)) + C && \text{Fundamental Theorem} \\
&= F(u) + C && u = g(x) \\
&= \int F'(u)\,du && \text{Fundamental Theorem} \\
&= \int f(u)\,du && F' = f
\end{aligned}$$

The Substitution Rule provides the following **substitution method** to evaluate the integral

$$\int f(g(x))g'(x)\,dx,$$

when f and g' are continuous functions:

1. Substitute $u = g(x)$ and $du = (du/dx)\,dx = g'(x)\,dx$ to obtain the integral

$$\int f(u)\,du.$$

2. Integrate with respect to u.
3. Replace u by $g(x)$ in the result.

EXAMPLE 3 Find $\int \sec^2(5t + 1) \cdot 5\,dt$.

Solution We substitute $u = 5t + 1$ and $du = 5\,dt$. Then,

$$\int \sec^2(5t+1)\cdot 5\,dt = \int \sec^2 u\,du \qquad \text{Let } u = 5t+1,\ du = 5\,dt.$$
$$= \tan u + C \qquad \frac{d}{du}\tan u = \sec^2 u$$
$$= \tan(5t+1) + C \qquad \text{Substitute } 5t+1 \text{ for } u.$$

EXAMPLE 4 Find $\int \cos(7\theta + 3)\,d\theta$.

Solution We let $u = 7\theta + 3$ so that $du = 7\,d\theta$. The constant factor 7 is missing from the $d\theta$ term in the integral. We can compensate for it by multiplying and dividing by 7, using the same procedure as in Example 2. Then,

$$\int \cos(7\theta+3)\,d\theta = \frac{1}{7}\int \cos(7\theta+3)\cdot 7\,d\theta \qquad \text{Place factor } 1/7 \text{ in front of integral.}$$
$$= \frac{1}{7}\int \cos u\,du \qquad \text{Let } u = 7\theta+3,\ du = 7\,d\theta.$$
$$= \frac{1}{7}\sin u + C \qquad \text{Integrate.}$$
$$= \frac{1}{7}\sin(7\theta+3) + C \qquad \text{Substitute } 7\theta+3 \text{ for } u.$$

There is another approach to this problem. With $u = 7\theta + 3$ and $du = 7\,d\theta$ as before, we solve for $d\theta$ to obtain $d\theta = (1/7)\,du$. Then the integral becomes

$$\int \cos(7\theta+3)\,d\theta = \int \cos u\cdot\frac{1}{7}\,du \qquad \text{Let } u = 7\theta+3,\ du = 7\,d\theta, \text{ and } d\theta = (1/7)\,du$$
$$= \frac{1}{7}\sin u + C \qquad \text{Integrate.}$$
$$= \frac{1}{7}\sin(7\theta+3) + C \qquad \text{Substitute } 7\theta+3 \text{ for } u.$$

We can verify this solution by differentiating and checking that we obtain the original function $\cos(7\theta + 3)$.