Republic of Iraq

Ministry of Higher Education and Scientific Research

Muthanna University

College of Education for Pure Science

Department of Mathematics

General Topology

Academic Year

2016-2017

Lecturer

Lect. Qays Hatem Imran

Chapter One

Topological Spaces

Definition: Let X be a non-empty set and let τ be a collection of subsets of X satisfying the following three conditions:

(i) $\phi \in \tau, X \in \tau$.

(ii) If $G_1 \in \tau$ and $G_2 \in \tau$, then $G_1 \cap G_2 \in \tau$.

(iii) If $G_\lambda \in \tau$ for every $\lambda \in \Lambda$ where Λ is an arbitrary set, then $\cup\{G_\lambda : \lambda \in \Lambda\} \in \tau$.

Then τ is called a topology for X, the members of τ are called τ-open (or simply open) sets and the pair (X, τ) is called a topological space. The elements of X will be called points of the space.

Remark: The union of empty collection of sets is empty. i.e $\cup\{A_\lambda : \lambda \in \phi\} = \phi$ and the intersection of empty collection of subsets of X is X itself. i.e $\cap\{A_\lambda : \lambda \in \phi\} = X$.

Remark: The three conditions (i), (ii) and (iii) are equivalent to the following two conditions:

(1) The intersection of an arbitrary finite number of open sets is open.

(2) The union of arbitrary collection of open sets is open.

Example: Let $X = \{a, b, c\}$ and consider the following collections of the subsets of X:

$\tau_1 = \{\phi, X\}$,

$\tau_2 = \{\phi, \{a\}, \{b, c\}, X\}$,

$\tau_3 = \{\phi, \{a\}, \{b\}, X\}$,

$\tau_4 = \{\phi, \{a\}, X\}$,

$\tau_5 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$,

$\tau_6 = \{\phi, \{a, b\}, X\}$,

$\tau_7 = \{\phi, \{b\}, \{a, b\}, X\}$,

$\tau_8 = \{\{a\}, \{b, c\}, X\}$,

$\tau_9 = \{\phi, \{a\}, \{b\}, \{a, b\}\}$,

$\tau_{10} = \{\phi, \{a, b\}, \{b, c\}, X\}$.

Then $\tau_1, \tau_2, \tau_4, \tau_5, \tau_6$ and τ_7 are all topologies for X, since they satisfy all the conditions (i), (ii) and (iii).

Let us verify these axioms for τ_7.

(i) $\phi, X \in \tau_7$

(ii) $\phi \cap \{b\} = \phi \cap \{a,b\} = \phi \cap X = \phi \in \tau_7$, $\{b\} \cap \{a,b\} = \{b\} \cap X = \{b\} \in \tau_7$ and $\{a,b\} \cap X = \{a,b\} \in \tau_7$.

(iii) $\phi \cup \{b\} = \{b\} \in \tau_7$, $\phi \cup \{a,b\} = \{a,b\} \in \tau_7$, $\phi \cup X = X \in \tau_7$, $\{b\} \cup \{a,b\} = \{a,b\} \in \tau_7$, $\{b\} \cup X = X \in \tau_7$, $\{a,b\} \cup X = X \in \tau_7$ and $\{b\} \cup \{a,b\} \cup X = X \in \tau_7$.

τ_3 is not a topology for X since $\{a\} \in \tau_3$ and $\{b\} \in \tau_3$ but $\{a\} \cup \{b\} = \{a,b\} \notin \tau_3$ and so it does not satisfy (iii).

τ_8 is not a topology for X since $\phi \notin \tau_8$ and so it does not satisfy (i).

τ_9 is not a topology for X since $X \notin \tau_9$ and so it does not satisfy (i).

τ_{10} is not a topology for X since $\{a,b\} \in \tau_{10}$ and $\{b,c\} \in \tau_{10}$ but $\{a,b\} \cap \{b,c\} = \{b\} \notin \tau_{10}$ and so it does not satisfy (ii).

Example: Let X be a non-empty set. Then the collection $I = \{\phi, X\}$ consisting of the empty set and the whole space is always a topology for X called the indiscrete (or trivial) topology. The pair (X, I) is called an indiscrete topological space.

Example: Let X be a non-empty set and let D be the collection of all subsets of X, the D is a topology for X called the discrete topology. The pair (X, D) is called a discrete topological space.

Solution: Since $\phi \subseteq X$, $X \subseteq X$, we have $\phi \in D$ and $X \in D$ so that (i) is satisfied, (ii) also holds since the intersection of two subsets of X a gain a subset of X. Similarly (iii) is satisfied since the union of any collection of subsets of X is a gain a subset of X.

Example: Let X be a non-empty set and let τ be the collection of all those subsets of X whose complements are finite together with the empty set, that is, a subset A of X belongs to τ iff A is empty or A^c is finite. Then τ is a topology for X called the co-finite topology or the finite complement topology.

Solution: (i) Since $X^c = \phi$ which is finite, we have $X \in \tau$. Also $\phi \in \tau$ be definition.

(ii) $G_1, G_2 \in \tau \Longrightarrow {G_1}^c, {G_2}^c$ are finite

$\Longrightarrow {G_1}^c \cup {G_2}^c$ is finite

$\Longrightarrow (G_1 \cap G_2)^c$ is finite [By De-Morgan law]

$\Longrightarrow G_1 \cap G_2 \in \tau$.

(iii) $G_\lambda \in \tau, \forall \lambda \in \Lambda \Longrightarrow {G_\lambda}^c$ is a finite, $\forall \lambda \in \Lambda$

$\Longrightarrow \cap \{{G_\lambda}^c : \lambda \in \Lambda\}$ is a finite

$\Longrightarrow [\cup \{G_\lambda : \lambda \in \Lambda\}]^c$ is a finite [By De-Morgan law]

$$\Longrightarrow \bigcup\{G_\lambda : \lambda \in \Lambda\} \in \tau.$$

Hence τ is co-finite topology for X.

Example: Let X be a non-empty set and let τ be consist of all those subsets of X whose complements are countable sets together with the empty set. Then τ is a topology for X called the co-countable topology.

Example: Let U consist of ϕ and all those subsets G of $\mathbb{R}$ having the property that to each $x \in G$ there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq G$. Then U is a topology for $\mathbb{R}$ called the usual topology.

Solution: (i) $\phi \in$ U by definition. Also $\mathbb{R} \in$ U since for each $x \in \mathbb{R}$, $(x-1, x+1) \subseteq \mathbb{R}$.

In fact for any $\varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \subseteq \mathbb{R}$.

(ii) Let $G_1, G_2 \in$ U. If $G_1 \cap G_2 = \phi$, there is nothing to prove.

If $G_1 \cap G_2 \neq \phi$, let $x \in G_1 \cap G_2$. Then $x \in G_1$ and $x \in G_2$. Hence there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $(x-\varepsilon_1, x+\varepsilon_1) \subseteq G_1$ and $(x-\varepsilon_2, x+\varepsilon_2) \subseteq G_2$. Take $\varepsilon = min\{\varepsilon_1, \varepsilon_2\}$. Then $\varepsilon > 0$ and $(x-\varepsilon, x+\varepsilon) \subseteq G_1 \cap G_2$. Hence $G_1 \cap G_2 \in$ U.

(iii) Let $\{G_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of members of U and let $x \in \bigcup\{G_\lambda : \lambda \in \Lambda\}$. Then $x \in G_\lambda$ for some $\lambda \in \Lambda$. Since $G_\lambda \in$ U, there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq G_\lambda$. But then $(x-\varepsilon, x+\varepsilon) \subseteq \bigcup\{G_\lambda : \lambda \in \Lambda\}$. Therefore $\bigcup\{G_\lambda : \lambda \in \Lambda\} \in$ U.

Hence U is a topology for $\mathbb{R}$.

Example: Every open interval on $\mathbb{R}$ is a U-open set.

Solution: Let (a, b) be any interval on $\mathbb{R}$ and let $x \in (a, b)$. Take $\varepsilon = min\{x-a, b-x\}$. Then it is easy to see that $(x-\varepsilon, x+\varepsilon) \subseteq (a, b)$. Hence (a, b) is a U-open set.

Example: Let τ_1 and τ_2 be the collections of subsets of $\mathbb{R}$ defined respectively as follows:

(1) $\phi \in \tau_1$, $\mathbb{R} \in \tau_1$ and all open infinite intervals $G_r = (r, \infty)$ with $r \in \mathbb{R}$ belong to τ_1.

(2) $\phi \in \tau_2$, $\mathbb{R} \in \tau_2$ and all open infinite intervals $G_q = (q, \infty)$ with $q \in \mathbb{Q}$ belong to τ_2.

Then τ_1 is a topology for $\mathbb{R}$ but τ_2 is not a topology for $\mathbb{R}$.

Solution: (1) (i) $\phi \in \tau_1$ and $\mathbb{R} \in \tau_1$ by definition.

(ii) Let $G_r \in \tau_1$ and $G_s \in \tau_1$ with $r, s \in \mathbb{R}$, $G_r \cap G_s = G_r \; or \; G_s$ according as $r \geq s \;\; or \;\; r \leq s$.

Hence $G_r \cap G_s \in \tau_1$.

(iii) Let $G_\lambda \in \tau_1$ for every $\lambda \in \Lambda$ where Λ is some set of real numbers. We have to show that $\bigcup\{G_\lambda : \lambda \in \Lambda\} \in \tau_1$.

If Λ is not bounded below so that $\inf(\Lambda) = -\infty$, then $\bigcup\{G_\lambda : \lambda \in \Lambda\} = \mathbb{R} \in \tau_1$.

If Λ is bounded below so that $\inf(\Lambda) = r_0 (r_0 \in \mathbb{R})$ exists, then $\bigcup\{G_\lambda : \lambda \in \Lambda\} = (r_0, \infty) = G_{r_0}$. Hence in either case, $\bigcup\{G_\lambda : \lambda \in \Lambda\} \in \tau_1$.

Therefore τ_1 is a topology for $\mathbb{R}$.

(2) Let G_q with $q \in \mathbb{Q}$ belong to τ_2 for all $q > \sqrt{2}$. Then $\bigcup\{G_q : q \in \mathbb{Q}, q > \sqrt{2}\} = (\sqrt{2}, \infty)$ which does not belong to τ_2, since $\sqrt{2}$ is not a rational number. Hence τ_2 is not a topology for $\mathbb{R}$.

Example: Let τ be the collection of subsets of $\mathbb{N}$ consisting of empty set and all subsets of $\mathbb{N}$ of the form $G_m = \{m, m+1, m+2, \ldots\ldots\}$, $m \in \mathbb{N}$. Then τ is a topology for $\mathbb{N}$.

Theorem: Let $\{\tau_\lambda : \lambda \in \Lambda\}$ where Λ is an arbitrary set, be a collection of topologies for X. Then the intersection $\bigcap\{\tau_\lambda : \lambda \in \Lambda\}$ is also a topology for X.

Proof: Let $\{\tau_\lambda : \lambda \in \Lambda\}$ be a collection of topologies on X. We have to show that $\bigcap\{\tau_\lambda : \lambda \in \Lambda\}$ is also a topology on X.

If $\Lambda = \phi$, then $\bigcap\{\tau_\lambda : \lambda \in \phi\} = P(X)$. Thus in this case the intersection of topologies is the discrete topology.

Now let $\Lambda \neq \phi$.

(i) Since τ_λ is a topology for every $\lambda \in \Lambda$, it follows that $\phi, X \in \tau_\lambda$ for every $\lambda \in \Lambda$.

But $\phi \in \tau_\lambda$ for every $\lambda \in \Lambda \Longrightarrow \phi \in \bigcap\{\tau_\lambda : \lambda \in \Lambda\}$, and

$X \in \tau_\lambda$ for every $\lambda \in \Lambda \Longrightarrow X \in \bigcap\{\tau_\lambda : \lambda \in \Lambda\}$.

(ii) Let $G_1, G_2 \in \bigcap\{\tau_\lambda : \lambda \in \Lambda\}$. Then $G_1, G_2 \in \tau_\lambda$ for every $\lambda \in \Lambda$. Since τ_λ is a topology for X for every $\lambda \in \Lambda$, it follows that $G_1 \cap G_2 \in \tau_\lambda$ for every $\lambda \in \Lambda$. Hence $G_1 \cap G_2 \in \bigcap\{\tau_\lambda : \lambda \in \Lambda\}$.

(iii) Let $G_\alpha \in \bigcap\{\tau_\lambda : \lambda \in \Lambda\}$ for all $\alpha \in \Delta$ where Δ is an arbitrary set. Then $G_\alpha \in \tau_\lambda$, $\forall \lambda \in \Lambda$ and $\forall \alpha \in \Delta$. Since each τ_λ is a topology for X, it follows that $\bigcup\{G_\alpha : \alpha \in \Delta\} \in \tau_\lambda$, $\forall \lambda \in \Lambda$. Hence $\bigcup\{G_\alpha : \alpha \in \Delta\} \in \bigcap\{\tau_\lambda : \lambda \in \Lambda\}$. Thus $\bigcap\{\tau_\lambda : \lambda \in \Lambda\}$ is a topology for X.

Remark: The union of topologies is not necessarily a topology on X.

Example: Let $X = \{a, b, c\}$. Consider two topologies τ_1 and τ_2 for X defined as follows:

$\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{b\}, X\}$. Then $\tau_1 \cup \tau_2 = \{\phi, \{a\}, \{b\}, X\}$ which is not a topology for X.

Definition: Let τ_1 and τ_2 be two topologies for a non-empty set X, we say that τ_1 is coarser (or weaker or smaller) than τ_2 or that τ_2 is finer (or stronger or larger) than τ_1 iff $\tau_1 \subseteq \tau_2$ that is iff every τ_1-open set is τ_2-open set.

If either $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$, then we say that τ_1 and τ_2 are comparable.
If $\tau_1 \nsubseteq \tau_2$ and $\tau_2 \nsubseteq \tau_1$, then we say that τ_1 and τ_2 are not comparable

Example: For a non-empty set X, the indiscrete topology I is the coarser topology and the discrete topology D is the finer topology.

Example: Let $X = \{a, b, c\}$. Consider three topologies τ_1, τ_2 and τ_3 for X defined as follows:
$\tau_1 = \{\phi, \{a\}, X\}$, $\tau_2 = \{\phi, \{b\}, X\}$ and $\tau_3 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$.
Since $\tau_1 \nsubseteq \tau_2$ and $\tau_2 \nsubseteq \tau_1$, then τ_1 and τ_2 are not comparable.
Since $\tau_2 \nsubseteq \tau_3$ and $\tau_3 \nsubseteq \tau_2$, then τ_2 and τ_3 are not comparable.
Since $\tau_1 \subseteq \tau_3$, then τ_1 and τ_3 are comparable.
Since $\tau_1 \subseteq \tau_3$, then τ_1 is coarser than τ_3 or τ_3 is finer than τ_1.

Metric topologies

Theorem: Let (X, d) be any metric space and let τ_d consist of ϕ and those subsets G of X having the property that to each $x \in G$ there exists $r > 0$ such that the open ball $B(x, r)$ is contained in G. Then τ_d is a topology for X.
Proof: (i) $\phi \in \tau_d$ by definition. Also to each $x \in X$, $B(x, 1) \subseteq X$, showing that $X \in \tau_d$.
(ii) Let $G_1, G_2 \in \tau_d$ and let $x \in G_1 \cap G_2$. Then $x \in G_1$ and $x \in G_2$. Hence there exist $r_1 > 0$ and $r_2 > 0$ such that $B(x, r_1) \subseteq G_1$ and $B(x, r_2) \subseteq G_2$.
Let $r = min\{r_1, r_2\}$. Then $B(x, r) \subseteq G_1 \cap G_2$ and therefore $G_1 \cap G_2 \in \tau_d$.
(iii) Let Ω be an arbitrary collection of members of τ_d and let $x \in \cup\Omega$. Then $x \in G$ for some $G \in \Omega$. Since $G \in \tau_d$, there exists $r > 0$ such that $B(x, r) \subseteq G$. But $B(x, r) \subseteq \cup\Omega$ and therefore $\cup\Omega \in \tau_d$. Hence τ_d is a topology for X.

Remark: Every open ball in a metric space (X, d) is an open set with respect to the d-metric topology for X.

Definition: A topological space (X, τ) is said to be metrizable iff there exists a metric d for X such that $\tau_d = \tau$. i.e d-metric topology for X is the same as τ.

Example: Let $X = \{a, b\}$, $a \neq b$. Define $\tau = \{\phi, \{a\}, X\}$. Then τ is a topology for X.
The topological space (X, τ) is not metrizable.
Solution: Let d be any metric space for X and let $d(a, b) = r$. Since $a \neq b, r > 0$.
Then $B(a, r) = \{a\} \in \tau$ and $B(b, r) = \{b\} \notin \tau$. Hence (X, τ) is not metrizable.

Example: Show that the usual metric for $\mathbb{R}$ induces usual topology for $\mathbb{R}$.

Example: Show that the discrete metric on a set X induces the discrete topology for X.

Closed sets

Definition: Let (X, τ) be a topological space. A subset F of X is said to be τ-closed if and only if its complement F^c is open.

Remark: Since ϕ is open, it follows that $\phi^c = X$ is closed. Similarly since X is open, $X^c = \phi$ is closed. Thus ϕ and X are open as well as closed in every topological space.

Example: Let $X = \{a, b, c\}$ and let $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ be a topology for X.
Solution: Since $\{a\}^c = \{b, c\}$, $\{b, c\}^c = \{a\}$, it follows that the closed sets are:
$X, \{b, c\}, \{a\}, \phi$.

Example: If $a \in \mathbb{R}$, then $\{a\}$ is a closed set in the usual topology for $\mathbb{R}$.
Solution: $\{a\}^c = (-\infty, a) \cup (a, \infty)$.
But $(-\infty, a)$ and (a, ∞) are U-open sets. Hence their union is also U-open.
It follows that $\{a\}^c$ is U-open. Therefore $\{a\}$ is U-closed.

Example: Let $a, b \in \mathbb{R}$ where $a < b$. Then the closed interval $[a, b]$ is closed set the usual topology for $\mathbb{R}$.

Solution: $[a, b]^c = \{x \in \mathbb{R} : x < a \text{ or } x > b\}$
$= (-\infty, a) \cup (b, \infty)$
which is U-open, being the union of two U-open sets. Hence $[a, b]$ is U-closed.

Definition: A topological space (X, τ) is said to be a door space iff every subset of X is either open or closed.

Example: Let $X = \{a, b, c\}$ and let $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}$.
Then closed sets are $X, \{a, c\}, \{c\}, \{a\}, \phi$. Hence all the subsets of X are either open or closed and consequently (X, τ) is a door space.

Theorem: If $\{F_\lambda : \lambda \in \Lambda\}$ is any collection of closed subsets of a topological space X, then $\bigcap\{F_\lambda : \lambda \in \Lambda\}$ is a closed set.
Proof: F_λ is closed, $\forall \lambda \in \Lambda \Longrightarrow {F_\lambda}^c$ is open, $\forall \lambda \in \Lambda$

$\Longrightarrow \bigcup\{F_\lambda{}^c : \lambda \in \Lambda\}$ is open [By part (iii) of the definition of topology]

$\Longrightarrow [\bigcap\{F_\lambda : \lambda \in \Lambda\}]^c$ is open [By De-Morgan law]

$\Longrightarrow \bigcap\{F_\lambda : \lambda \in \Lambda\}$ is a closed [By definition of closed sets].

Theorem: If F_1 and F_2 are two closed subsets of a topological space (X,τ). Then $F_1 \cup F_2$ is a closed set.

Proof: F_1, F_2 are closed $\Longrightarrow F_1{}^c, F_2{}^c$ are open

$\Longrightarrow F_1{}^c \cap F_2{}^c$ is open [By part (ii) of the definition of topology]

$\Longrightarrow [F_1 \cup F_2]^c$ is open [By De-Morgan law]

$\Longrightarrow F_1 \cup F_2$ is closed.

Remark: Note that if $F_1, F_2, \ldots\ldots, F_n$ be a finite number of closed subsets of X, then their union will also be a closed subset of X.

Remark: The union of an infinite collection of closed sets in a topological space is not necessarily closed.

Example: Let $(\mathbb{R}, \mathrm{U})$ be the usual topological space and let $F_n = [\frac{1}{n}, 1]$, $n \in \mathbb{N}$ so that F_n is a closed interval on $\mathbb{R}$.

Solution: F_n is a U-closed set. Now

$$\bigcup\{F_n : n \in \mathbb{N}\} = \{1\} \cup [\tfrac{1}{2}, 1] \cup [\tfrac{1}{3}, 1] \cup [\tfrac{1}{4}, 1] \ldots\ldots = (0,1]$$

Since $(0,1]$ is not closed, it follows that the union of an infinite collection of closed sets is not necessarily closed.

Example: Every finite subset of $\mathbb{R}$ is a U-closed set.

Theorem: Let X be a non-empty set and F be a family of subsets of X such that

(i) $\phi \in \mathrm{F}$ and $X \in \mathrm{F}$

(ii) $F_1, F_2 \in \mathrm{F} \Longrightarrow F_1 \cup F_2 \in \mathrm{F}$.

(iii) $F_\lambda \in \mathrm{F}, \forall \lambda \in \Lambda \Longrightarrow \bigcap\{F_\lambda : \lambda \in \Lambda\} \in \mathrm{F}$.

Then there exists a unique topology for X such that the τ-closed subsets of X are precisely the members of F.

Proof: H.W.

Neighbourhoods

Definition: Let (X,τ) be a topological space and let $x \in X$. A subset N of X is said to be a τ-neighbourhood (briefly τ-nhd) of x iff there exists a τ-open set G such that $x \in G \subseteq N$.
Similarly N is called a τ-nhd of $A \subseteq X$ iff there exists an open set G such that $A \subseteq G \subseteq N$.
The collection of all τ-neighbourhoods of $x \in X$ is called the neighbourhood system at x and shall be denoted by $N(x)$.

Remark: (i) τ-open set is a τ-neighbourhood of each of its points.
(ii) τ-neighbourhood of a point need not be a τ-open set.
(iii) every open set containing x is a nhd of x.

Example: Let $X = \{1,2,3\}$ and let $\tau = \{\phi, \{1\}, \{2,3\}, X\}$ be a topology for X. Find the nhd system of 1,2,3.
Solution: Then all subsets of X are: $\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}$ and X.
$N(1) = \{\{1\}, \{1,2\}, \{1,3\}, X\}$,
$N(2) = \{\{2,3\}, X\}$,
$N(3) = \{\{2,3\}, X\}$.

Example: Let $X = \{1,2,3,4,5\}$ and let $\tau = \{\phi, \{1\}, \{1,2\}, \{1,2,5\}, \{1,3,4\}, \{1,2,3,4\}, X\}$ be a topology for X. Find the nhd system of 1,2,3,4,5.
Solution: H.W.

Theorem: A subset of a topological space is open if and only if it is a neighbourhood of each of its points.
Proof: Let G be an open subset of a topological space. Then for every $x \in G$ such that $x \in G \subseteq G$ and therefore G is a nhd of each of its points.
Conversely, let G be a nhd of each of its points. If $G = \phi$, then it is open.
If $G \neq \phi$, then to each $x \in G$ there exists an open set G_x such that $x \in G_x \subseteq G$. It follows that $G = \bigcup\{G_x : x \in G\}$. Hence G is open, being a union of open sets.

Properties of neighbourhoods

Theorem: Let X be a topological space, and for each $x \in X$, let $N(x)$ be the collection of all nhds of x. Then:
(1) $\forall x \in X, N(x) \neq \phi$
i.e every point x has at least one neighbourhood.

(2) $N \in N(x) \implies x \in N$

i.e every neighbourhood of x contains x.

(3) $N \in N(x), N \subseteq M \implies M \in N(x)$

i.e every set containing a neighbourhood of x is a neighbourhood of x.

(4) $N \in N(x), M \in N(x) \implies N \cap M \in N(x)$

i.e the intersection of two neighbourhoods of x is a neighbourhood of x.

(5) $N \in N(x) \implies \exists M \in N(x)$ such that $M \subseteq N$ and $M \in N(y)\ \forall y \in M$

i.e if N is a neighbourhood of x, then there exists a neighbourhood M of x which is a subset of N such that M is a neighbourhood of each of its points.

Proof: (1) Since X is an open set, it is a nhd of every $x \in X$. Hence there exists at least one nhd (namely X) for each $x \in X$. Hence $N(x) \neq \phi, \forall x \in X$.

(2) If $N \in N(x)$, then N is a nhd of x. So by definition of nhd, $x \in N$.

(3) If $N \in N(x)$, then there is an open set G such that $x \in G \subseteq N$. Since $N \subseteq M$, $x \in G \subseteq M$ and so M is a nhd of x. Hence $M \in N(x)$.

(4) Let $N \in N(x)$ and $M \in N(x)$. Then by definition of nhd, there exist open sets G_1 and G_2 such that $x \in G_1 \subseteq N$ and $x \in G_2 \subseteq M$. Hence $x \in G_1 \cap G_2 \subseteq N \cap M$………… (1)

Since $G_1 \cap G_2$ is open set, it follows from (1) that $N \cap M$ is a nhd of x. Hence $N \cap M \in N(x)$.

(5) If $N \in N(x)$, then there exists an open set M such that $x \in M \subseteq N$. Since M is open set, it is a nhd of each of its points. Therefore $M \in N(y)\ \forall y \in M$.

Theorem: Let X be a non-empty set and for each $x \in X$, let $N(x)$ be a non-empty collection of subsets of X satisfying the following conditions:

(1) $N \in N(x) \implies x \in N$

(2) $N \in N(x), M \in N(x) \implies N \cap M \in N(x)$.

Let τ consist of the empty set and all those non-empty subsets G of X having the property that $x \in G$ implies that there exists an $N \in N(x)$ such that $x \in N \subseteq G$. Then τ is a topology for X.

Proof: (i) $\phi \in \tau$ by definition. We now show that $X \in \tau$.

Let $x \in X$. Since $N(x) \neq \phi$, there is an $N \in N(x)$ and so $x \in N$ by (1). Since N is a subset of X, we have $x \in N \subseteq X$. Hence $X \in \tau$.

(ii) Let $G_1, G_2 \in \tau$. If $x \in G_1 \cap G_2$, then $x \in G_1$ and $x \in G_2$. Since $G_1 \in \tau$ and $G_2 \in \tau$, there exist $N \in N(x)$ and $M \in N(x)$ such that $x \in N \subseteq G_1$ and $x \in M \subseteq G_2$. Then $x \in N \cap M \subseteq G_1 \cap G_2$. But $N \cap M \in N(x)$ by (2). Hence $G_1 \cap G_2 \in \tau$.

(iii) Let $G_\lambda \in \tau, \forall\lambda \in \Lambda$. If $x \in \cup\{G_\lambda : \lambda \in \Lambda\}$, then $x \in G_{\lambda_x}$ for some $\lambda_x \in \Lambda$. Since $G_{\lambda_x} \in \tau$, there exists an $N \in N(x)$ such that $x \in N \subseteq G_{\lambda_x}$ and consequently $x \in N \subseteq \cup\{G_\lambda : \lambda \in \Lambda\}$. Hence $\cup\{G_\lambda : \lambda \in \Lambda\} \in \tau$. It follows that τ is a topology for X.

Definition: Let (X,τ) be a topological space. A non-empty collection $\beta(x)$ of τ-nhds of x is called a base for the τ-neighbourhood system of x iff for every τ-neighbourhood N of x there is $B \in \beta(x)$ such that $B \subseteq N$. We then also say that $\beta(x)$ is a local base at x or a fundamental system of neighbourhoods of x.
If $\beta(x)$ is a local base at x, then the members of $\beta(x)$ are called basic τ-neighbourhoods of x.

Example: Let $X = \{a,b,c,d,e\}$ and let $\tau = \{\phi,\{a\},\{a,b\},\{a,c,d\},\{a,b,c,d\},\{a,b,e\},X\}$ be a topology for X.
Then a local base at each of the points a,b,c,d,e is given by:

$$\beta(a) = \{\{a\}\}, \quad \beta(b) = \{\{a,b\}\}, \quad \beta(c) = \{\{a,c,d\}\},$$
$$\beta(d) = \{\{a,c,d\}\}, \quad \beta(e) = \{\{a,b,e\}\}.$$

Observe that here a local base at each point consists of a single τ-nhd of the point.
Note that $\{\{a,b\}\}$ does not form a local base at a. (Why ?)

Example: Consider the usual topology U for $\mathbb{R}$ and any point $x \in \mathbb{R}$. Then the collection $\beta(x) = \{(x-\varepsilon, x+\varepsilon): 0 < \varepsilon \in \mathbb{R}\}$ constitutes a base for the U-neighbourhood system of x.
Solution: Let N be any nhd of x. Then there exists a U-open set G such that $x \in G \subseteq N$.
Since G is U-open, $\exists\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq G \subseteq N$. Thus to each nhd N of x, $\exists$ a member $(x-\varepsilon, x+\varepsilon)$ of $\beta(x)$ such that $(x-\varepsilon, x+\varepsilon) \subseteq N$.

Definition: A topological space (X,τ) is said to satisfy the first axiom of countability if each point of X possesses a countable local base. Such a topological space is said to be a first countable space.

Example: A discrete topological space (X,D) is first countable space.

Example: The usual topological space $(\mathbb{R},\text{U})$ is first countable space.
Solution: Let $x \in \mathbb{R}$. Then the collection $\{(x-\frac{1}{n}, x+\frac{1}{n}) : n \in N\}$ is a countable base at x and so $(\mathbb{R},\text{U})$ is first countable.

Properties of local base

Theorem: Let (X,τ) be a topological space and let $\beta(x)$ be a local base at any point x of X. Then $\beta(x)$ has the following properties:

(1) $\beta(x) \neq \phi$ for every $x \in X$

(2) If $B \in \beta(x)$, then $x \in B$

(3) If $A \in \beta(x)$ and $B \in \beta(x)$, then there exists a $C \in \beta(x)$ such that $C \subseteq A \cap B$

(4) If $A \in \beta(x)$, then there exists a set B such that $x \in B \subseteq A$ and $\forall y \in B$, $\exists C \in \beta(y)$ satisfying $C \subseteq B$.

Definition: Let (X,τ) be a topological space. A collection β of subsets of X is said to form a base for τ if and only if

(i) $\beta \subseteq \tau$,

(ii) for each point $x \in X$ and each nhd N of x there exists some $B \in \beta$ such that $x \in B \subseteq N$.

Example: Let $X = \{a,b,c,d\}$ and let $\tau = \{\phi,\{a\},\{b\},\{a,b\},\{c,d\},\{a,c,d\},\{b,c,d\},X\}$ be a topology for X.

Then the collection $\beta = \{\{a\},\{b\},\{c,d\}\}$ is a base for τ since

(i) $\beta \subseteq \tau$ and

(ii) Each nhd of a contains $\{a\}$ with a member of β containing a. Similarly each nhd of b contains $\{b\} \in \beta$ and each nhd of c or d contains $\{c,d\} \in \beta$.

Definition: Let (X,τ) be a topological space. The space is said to be second countable iff there exists a countable base for τ.

Example: The usual topological space $(\mathbb{R},\mathrm{U})$ is second countable space.

Theorem: Let (X,τ) be a topological space. A sub collection β of τ is a base for τ iff every τ-open set can be expressed as the union of members of β.

Proof: Let β be a base for τ and let $G \in \tau$. Since G is τ-open, it is a τ-nhd of each of its points. Hence by definition of base, $\forall x \in G$, $\exists B \in \beta$ such that $x \in B \subseteq G$. It follows that $G = \bigcup\{B : B \in \beta \text{ and } B \subseteq G\}$.

Conversely, let $\beta \subseteq \tau$ and let every open set G be the union of members of β. We have to show that β is a base for τ. We have

(i) $\beta \subseteq \tau$ (given)

(ii) Let $x \in X$ and let N be any nhd of x. Then there exists an open set G such that $x \in G \subseteq N$. But G is the union of members of β. Hence there exists $B \in \beta$ such that $x \in B \subseteq G \subseteq N$. Thus β is a base for τ.

Theorem: Let τ and τ' be topologies for X which have a common base β. Then $\tau = \tau'$.

Proof: Let $G \in \tau$ and $x \in G$. Since G is τ-open, it is a τ-nhd of x and since β is a base for τ, there exists $B \in \beta$ such that $x \in B \subseteq G$. Since β is a base for τ' and $B \in \beta$, it follows that $B \in \tau'$. Hence G is τ'-nhd of x. Since x is arbitrary, $G \in \tau'$.
Thus $\tau \subseteq \tau'$. By symmetry $\tau' \subseteq \tau$. Hence $\tau = \tau'$.

Properties of a base for a topology

Theorem: Let (X, τ) be a topological space and let β be a base for τ. Then β has the following properties:

(1) For every $x \in X$, there exists a $B \in \beta$ such that $x \in B$, that is, $X = \bigcup\{B : B \in \beta\}$.

(2) For every $B_1, B_2 \in \beta$ and every point $x \in B_1 \cap B_2$ there exists a $B \in \beta$ such that $x \in B \subseteq B_1 \cap B_2$.

Theorem: Let X be a non-empty set and let β be a collection of subsets of X satisfying the following conditions:

(1) For every $x \in X$, there exists a $B \in \beta$ such that $x \in B$, that is, $X = \bigcup\{B : B \in \beta\}$.

(2) For every $B_1, B_2 \in \beta$ and every point $x \in B_1 \cap B_2$ there exists a $B \in \beta$ such that $x \in B \subseteq B_1 \cap B_2$.

Then there exists a unique topology τ for X such that β is a base for τ.

Definition: Let (X, τ) be a topological space. A collection β_* of subsets of X is called a sub-base for the topology τ iff $\beta_* \subseteq \tau$ and finite intersections of members of β_* form a base for τ.

Example: Let $X = \{a, b, c, d\}$ and let $\tau = \{\phi, \{a\}, \{a, c\}, \{a, d\}, \{a, c, d\}, X\}$.
Then $\beta_* = \{\{a, c\}, \{a, d\}, X\}$ is a sub-base for τ. Since the family β of finite intersections of members of β_* is given by $\beta = \{\{a\}, \{a, c\}, \{a, d\}, X\}$ which is a base for τ.

Derived sets

Definition: Let (X, τ) be a topological space and let A be a subset of X. A point $x \in X$ is called a limit point (or a cluster point) of A iff every neighbourhood of x contains a point of A

other than x. The set of all limit points of A is called the derived set of A and denoted by $D_X(A)$ or $D(A)$. i.e $(N-\{x\})\cap A \neq \phi$, for every τ-nhd N of x.

Example: Let $X=\{1,2,3\}$ and let $\tau=\{\phi,\{1\},\{2,3\},X\}$. Find all the limit points of the set $A=\{1,2\}$.

Solution: $N(1)=\{\{1\},\{1,2\},\{1,3\},X\}$.

$(\{1\}-\{1\})\cap\{1,2\}=\phi\cap\{1,2\}=\phi$.

$\therefore$ 1 is not limit point of A.

$N(2)=\{\{2,3\},X\}$.

$(\{2,3\}-\{2\})\cap\{1,2\}=\{3\}\cap\{1,2\}=\phi$.

$\therefore$ 2 is not limit point of A.

$N(3)=\{\{2,3\},X\}$.

$(\{2,3\}-\{3\})\cap\{1,2\}=\{2\}\cap\{1,2\}=\{2\}\neq\phi$.

$\therefore$ 3 is limit point of A.

Hence $D(A)=\{3\}$.

Example: Let $X=\{a,b,c\}$ and let $\tau=\{\phi,\{a\},\{a,b\},\{a,c\},X\}$. Find all the limit points of the sets (i) $A=\{b,c\}$ (ii) $B=\{a,c\}$.

Solution: $N(a)=\{\{a\},\{a,b\},\{a,c\},X\}$, $N(b)=\{\{a,b\},X\}$ and $N(c)=\{\{a,c\},X\}$.

(i) $A=\{b,c\}$

$(\{a\}-\{a\})\cap\{b,c\}=\phi\cap\{b,c\}=\phi$.

$\therefore$ a is not limit point of A.

$(\{a,b\}-\{b\})\cap\{b,c\}=\{a\}\cap\{b,c\}=\phi$.

$\therefore$ b is not limit point of A.

$(\{a,c\}-\{c\})\cap\{b,c\}=\{a\}\cap\{b,c\}=\phi$.

$\therefore$ c is not limit point of A.

Hence $D(A)=\phi$.

(ii) $B=\{a,c\}$.

$(\{a\}-\{a\})\cap\{a,c\}=\phi\cap\{a,c\}=\phi$.

$\therefore$ a is not limit point of B.

$(\{a,b\}-\{b\})\cap\{a,c\}=\{a\}\cap\{a,c\}=\{a\}\neq\phi$.

$\therefore$ b is limit point of B.

$(\{a,c\}-\{c\})\cap\{a,c\}=\{a\}\cap\{a,c\}=\{a\}\neq\phi$.

$\therefore$ c is limit point of B.

Hence $D(B) = \{b, c\}$.

Definition: Let A be a subset of a topological space (X, τ) and let $x \in X$. Then x is called an adherent point of A iff every nhd of x contains a point of A. The set of all adherent points of A is called the adherence of A and denoted by $Adh(A)$. i.e $N \cap A \neq \phi$, for every nhd N of x.

Example: Let $X = \{1,2,3\}$ and let $\tau = \{\phi, \{1\}, \{2,3\}, X\}$. Find all the adherent points of the set $A = \{1,2\}$.

Solution: $N(1) = \{\{1\}, \{1,2\}, \{1,3\}, X\}$, $N(2) = \{\{2,3\}, X\}$ and $N(3) = \{\{2,3\}, X\}$.

$\{1\} \cap \{1,2\} = \{1\} \neq \phi$, $\{1,2\} \cap \{1,2\} = \{1,2\} \neq \phi$, $\{1,3\} \cap \{1,2\} = \{1\} \neq \phi$ and

$X \cap \{1,2\} = \{1,2\} \neq \phi$.

$\therefore$ 1 is adherent point of A.

$\{2,3\} \cap \{1,2\} = \{2\} \neq \phi$ and $X \cap \{1,2\} = \{1,2\} \neq \phi$.

$\therefore$ 2 is adherent point of A.

$\{2,3\} \cap \{1,2\} = \{2\} \neq \phi$ and $X \cap \{1,2\} = \{1,2\} \neq \phi$.

$\therefore$ 3 is adherent point of A.

The adherent points of A are 1,2,3. Hence $Adh(A) = \{1,2,3\}$.

Theorem: Let (X, τ) be a topological space and let A be a subset of X. Then A is closed if and only if $D(A) \subseteq A$.

Proof: Let A be a closed. Then A^c is open and to each $x \in A^c$ there exists a nhd N_x of x such that $N_x \subseteq A^c$. Since $A \cap A^c = \phi$, the nhd N_x contains no point of A and so x is not a limit point of A. Thus no point of A^c can be a limit point of A, that is, A contains all its limit points. Hence $D(A) \subseteq A$.

Conversely, let $D(A) \subseteq A$ and let $x \in A^c$. Then $x \notin A$. Since $D(A) \subseteq A$, $x \notin D(A)$. Hence there exists a nhd N_x of x such that $N_x \cap A = \phi$ so that $N_x \subseteq A^c$. Thus A^c contains a nhd of each of its points and so A^c is open, that is, A is closed.

Properties of derived sets

Theorem: Let A, B be subsets of a topological space (X, τ). Then:

(i) $D(\phi) = \phi$.

(ii) $A \subseteq B \Longrightarrow D(A) \subseteq D(B)$.

(iii) $D(A \cap B) \subseteq D(A) \cap D(B)$.

(iv) $D(A \cup B) = D(A) \cup D(B)$.

Proof: (i) Since ϕ is closed, $D(\phi) \subseteq \phi$. But ϕ is a subset of every set and so $\phi \subseteq D(\phi)$. Hence $D(\phi) = \phi$.

(ii) Let $p \in D(A)$ so that p is a limit point of A. Then every nhd of p contains a point of A different from p. Since $A \subseteq B$, every nhd of p must also contain a point of B different from p. Hence p is also a limit point of B, that is, $p \in D(B)$. Hence $D(A) \subseteq D(B)$.

(iii) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (ii) we have $D(A \cap B) \subseteq D(A)$ and $D(A \cap B) \subseteq D(B)$. Hence $D(A \cap B) \subseteq D(A) \cap D(B)$.

(iv) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, it follows from (ii) that $D(A) \subseteq D(A \cup B)$ and $D(B) \subseteq D(A \cup B)$ and hence $D(A) \cup D(B) \subseteq D(A \cup B)$.

Conversely, let $x \notin D(A) \cup D(B) \Longrightarrow x \notin D(A \cup B)$.

If $x \notin D(A) \cup D(B)$, then $x \notin D(A)$ and $x \notin D(B)$, that is, x is neither a limit point of A nor a limit point of B. Hence there exist nhds N_1 and N_2 of x such that $(N_1 - \{x\}) \cap A = \phi$ and $(N_2 - \{x\}) \cap B = \phi$…………… (1)

Now $N = N_1 \cap N_2$ is a nhd of x which by (1) contains no point of $A \cup B$ except x. It follows that $x \notin D(A \cup B)$ as required. Hence $D(A \cup B) \subseteq D(A) \cup D(B)$.

Thus $D(A \cup B) = D(A) \cup D(B)$.

Closure

Definition: Let (X, τ) be a topological space and let A be a subset of X. Then the intersection of all τ-closed containing the set A is called the closure of A and denoted by $\bar{A}$ or $c(A)$ or $cl(A)$. i.e $cl(A) = \bigcap\{F: F \text{ is closed}, A \subseteq F\}$.

Example: Let $X = \{a, b, c, d\}$ and let $\tau = \{\phi, \{a\}, \{a, c\}, \{a, d\}, \{a, c, d\}, X\}$. Find the closure of the sets (i) $\{b, c\}$ (ii) $\{b\}$ (iii) $\{b, c, d\}$.

Solution: The closed subsets of X are $X, \{b, c, d\}, \{b, d\}, \{b, c\}, \{b\}$ and ϕ.

$cl(A) = \bigcap\{F: F \text{ is closed}, A \subseteq F\}$.

(i) $cl(\{b, c\}) = X \cap \{b, c, d\} \cap \{b, c\} = \{b, c\}$.

(ii) $cl(\{b\}) = X \cap \{b, c, d\} \cap \{b, d\} \cap \{b, c\} \cap \{b\} = \{b\}$.

(iii) $cl(\{b, c, d\}) = X \cap \{b, c, d\} = \{b, c, d\}$.

Theorem: Let A be a subset of a topological space (X, τ). Then:

(i) $cl(A)$ is the smallest closed set containing A.

(ii) A is closed iff $cl(A) = A$.

Proof: (i) This follows from definition.

(ii) If A is closed, then A itself is the smallest closed set containing A and hence $cl(A) = A$. Conversely, let $cl(A) = A$. By (i), $cl(A)$ is closed and so A is also closed.

Theorem: $cl(A) = A \cup D(A)$.

Proof: H.W.

Corollary: $cl(A) = Adh(A) = \{x: \text{each nhd of } x \text{ intersects } A\}$.

Proof: $x \in Adh(A) \Leftrightarrow$ every nhd of x intersects A

$\Leftrightarrow x \in A$ or every nhd of x contains a point of A other than x.

$\Leftrightarrow x \in A$ or $x \in D(A)$

$\Leftrightarrow x \in A \cup D(A)$

$\Leftrightarrow x \in cl(A)$.

Example: Consider the co-finite topological space (X, τ) and find the closure of any subset A of X.

Solution: Here τ consists of the empty set ϕ and all those subsets of X whose complements are finite so that the closed subsets of X are all the finite subsets of X together with X. Hence if $A \subseteq X$ is finite, its closure $cl(A)$ is A itself since A is closed and if A is infinite then the only closed super set of A is X and so $cl(A) = X$. Thus $cl(A) = A$ if A is finite and $cl(A) = X$ if A is infinite.

Properties of closure

Theorem: Let (X, τ) be a topological space and let A, B be any subsets of X. Then:

(i) $cl(\phi) = \phi$, $cl(X) = X$.

(ii) $A \subseteq cl(A)$.

(iii) $A \subseteq B \Longrightarrow cl(A) \subseteq cl(B)$.

(iv) $cl(A \cup B) = cl(A) \cup cl(B)$.

(v) $cl(A \cap B) \subseteq cl(A) \cap cl(B)$.

(vi) $cl\big(cl(A)\big) = cl(A)$.

Proof: (i) Since ϕ is closed, we have $cl(\phi) = \phi$.

Since X is closed, we have $cl(X) = X$.

(ii) By theorem (i), $cl(A)$ is the smallest closed set containing A and so $A \subseteq cl(A)$.

(iii) By part (ii), $B \subseteq cl(B)$. Since $A \subseteq B$, we have $A \subseteq cl(B)$. But $cl(B)$ is a closed set. Thus $cl(B)$ is a closed set containing A. Since $cl(A)$ is the smallest closed set containing A, we have $cl(A) \subseteq cl(B)$. Hence $A \subseteq B \Longrightarrow cl(A) \subseteq cl(B)$.

(iv) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, we have $cl(A) \subseteq cl(A \cup B)$ and $cl(B) \subseteq cl(A \cup B)$ by part (iii). Hence $cl(A) \cup cl(B) \subseteq cl(A \cup B)$………… (1)

Since $cl(A)$ and $cl(B)$ are closed sets, $cl(A) \cup cl(B)$ is also closed. Also $A \subseteq cl(A)$ and $B \subseteq cl(B)$ implies that $A \cup B \subseteq cl(A) \cup cl(B)$. Thus $cl(A) \cup cl(B)$ is a closed set containing $A \cup B$. Since $cl(A \cup B)$ is the smallest closed set containing $A \cup B$, we have

$cl(A \cup B) \subseteq cl(A) \cup cl(B)$………… (2)

From (1) and (2), we have $cl(A \cup B) = cl(A) \cup cl(B)$.

(v) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by part (iii) we have $cl(A \cap B) \subseteq cl(A)$ and $cl(A \cap B) \subseteq cl(B)$. Hence $cl(A \cap B) \subseteq cl(A) \cap cl(B)$.

(vi) Since $cl(A)$ is a closed set, we have $cl(cl(A)) = cl(A)$ by theorem (ii) [A is closed iff $cl(A) = A$].

Theorem: Let (X, τ) be a topological space and let A be subset of X. Then the following statements are equivalent:

(i) A is closed.

(ii) $cl(A) = A$.

(iii) A contains all its limit points.

Proof: (i) $\Longrightarrow$ (ii) : A is closed $\Longrightarrow cl(A) = A$. By theorem part (ii) [$A$ is closed iff $cl(A) = A$]

(ii) $\Longrightarrow$ (iii) : $cl(A) = A \Longrightarrow A \cup D(A) = A$. By theorem [$cl(A) = A \cup D(A)$]

$\Longrightarrow D(A) \subseteq A \Longrightarrow$ A contains all its limit points.

(iii) $\Longrightarrow$ (i) : A contains all its limit points $\Longrightarrow D(A) \subseteq A$

$\Longrightarrow A \cup D(A) = A$

$\Longrightarrow cl(A) = A$

$\Longrightarrow$ A is closed. By theorem part (ii)

Interior of a set

Definition: Let (X, τ) be a topological space and let A be a subset of X. A point $x \in X$ is said to be an interior point of A iff A is a neighbourhood of x, that is, iff there exists an open set G such that $x \in G \subseteq A$. The set all interior points of A is called the interior of A and is denoted by A° or A^{i} or $i(A)$ or $int(A)$.

Example: Let $X = \{a, b, c, d\}$ and let $\tau = \{\phi, \{a\}, \{a, c\}, \{a, d\}, \{a, c, d\}, X\}$. Find the interior points of the sets (i) $A = \{b, c\}$ (ii) $B = \{a, d\}$

Solution: (i) $a \in \{a\} \nsubseteq \{b, c\}$, $a \in \{a, c\} \nsubseteq \{b, c\}$, $a \in \{a, d\} \nsubseteq \{b, c\}$, $a \in \{a, c, d\} \nsubseteq \{b, c\}$ and $a \in X \nsubseteq \{b, c\}$.

$\therefore a$ is not interior point of A.

$b \in X \nsubseteq \{b,c\}$

$\therefore b$ is not interior point of A.

$c \in \{a,c\} \nsubseteq \{b,c\}$, $c \in \{a,c,d\} \nsubseteq \{b,c\}$ and $c \in X \nsubseteq \{b,c\}$.

$\therefore c$ is not interior point of A.

$d \in \{a,d\} \nsubseteq \{b,c\}$, $d \in \{a,c,d\} \nsubseteq \{b,c\}$ and $d \in X \nsubseteq \{b,c\}$.

$\therefore d$ is not interior point of A.

$int(A) = \phi$.

(ii) $a \in \{a\} \subseteq \{a,d\}$

$\therefore a$ is interior point of B.

$b \in X \nsubseteq \{a,d\}$

$\therefore b$ is not interior point of B.

$c \in \{a,c\} \nsubseteq \{a,d\}$, $c \in \{a,c,d\} \nsubseteq \{a,d\}$ and $c \in X \nsubseteq \{a,d\}$.

$\therefore c$ is not interior point of B.

$d \in \{a,d\} \subseteq \{a,d\}$

$\therefore d$ is interior point of B.

$int(B) = \{a,d\}$.

Theorem: $int(A) = \cup\{G: G \text{ is open}, G \subseteq A\}$.

Proof: $x \in int(A) \Leftrightarrow A$ is a nhd of x

$\Leftrightarrow$ there exists an open set G such that $x \in G \subseteq A$

$\Leftrightarrow x \in \cup\{G: G \text{ is open}, G \subseteq A\}$.

Hence, $int(A) = \cup\{G: G \text{ is open}, G \subseteq A\}$.

Theorem: Let (X,τ) be a topological space and let A be a subset of X. Then:

(i) $int(A)$ is an open set.

(ii) $int(A)$ is the largest open set contained in A.

(iii) A is an open set iff $int(A) = A$.

Proof: (i) Let $x \in int(A)$. Then x is an interior point of A. Hence by definition, A is a nhd of x. Then there exists an open set G such that $x \in G \subseteq A$. Since G is open, it is a nhd of each of its points and so A is also a nhd of each point of G. It follows that every point of G is an interior point of A so that $G \subseteq int(A)$. Thus it is shown that to each $x \in int(A)$, there exists an open set G such that $x \in G \subseteq int(A)$. Hence $int(A)$ is a nhd of each of its points and consequently $int(A)$ is open set.

(ii) Let G be any open subset of A and let $x \in G$ so that $x \in G \subseteq A$. Since G is open, A is a nhd of x and consequently x is an interior point of A. Hence $x \in int(A)$. Thus we have shown that $x \in G \Longrightarrow x \in int(A)$ and so $G \subseteq int(A) \subseteq A$. Hence $int(A)$ contains every open subset of A and it is therefore the largest open subset of A.

(iii) Let $int(A) = A$. By part (i) $int(A)$ is an open set and therefore A is also open.
Conversely, let A be open set. Then A is surely identical with the largest open subset of A.
But by part (ii), $int(A)$ is the largest open subset of A. Hence $int(A) = A$.

Properties of interior

Theorem: Let (X, τ) be a topological space and let A, B be any subsets of X. Then:

(i) $int(\phi) = \phi$, $int(X) = X$.

(ii) $int(A) \subseteq A$.

(iii) $A \subseteq B \Longrightarrow int(A) \subseteq int(B)$.

(iv) $int(A \cap B) = int(A) \cap int(B)$.

(v) $int(A) \cup int(B) \subseteq int(A \cup B)$.

(vi) $int\big(int(A)\big) = int(A)$.

Proof: (i) Since ϕ and X are open sets, we have by part (iii) of theorem [A is an open set iff $int(A) = A$], $int(\phi) = \phi$, $int(X) = X$.

(ii) $x \in int(A) \Longrightarrow x$ is an interior point of A

$\Longrightarrow A$ is a nhd of $x \Longrightarrow x \in A$.

Hence $int(A) \subseteq A$.

(iii) Let $x \in int(A)$. Then x is an interior point of A and so A is a nhd of x. Since $A \subseteq B$, B is also a nhd of x. This implies that $x \in int(B)$. Hence $int(A) \subseteq int(B)$.

(iv) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by part (iii) we have $int(A \cap B) \subseteq int(A)$ and $int(A \cap B) \subseteq int(B)$. Hence $int(A \cap B) \subseteq int(A) \cap int(B)$………… (1)

Let $x \in int(A) \cap int(B)$. Then $x \in int(A)$ and $x \in int(B)$. Hence x is an interior point of each of the sets A and B. It follows that A and B are nhds of x so that $A \cap B$ is also a nhd of x. Hence $x \in int(A \cap B)$. Hence $int(A) \cap int(B) \subseteq int(A \cap B)$……….. (2)

From (1) and (2), we get $int(A \cap B) = int(A) \cap int(B)$.

(v) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, we have $int(A) \subseteq int(A \cup B)$ and $int(B) \subseteq int(A \cup B)$ by part (iii). Hence $int(A) \cup int(B) \subseteq int(A \cup B)$.

(vi) By (i) of theorem, $int(A)$ is an open set. Hence by part (iii) of the same theorem $int\big(int(A)\big) = int(A)$.

Exterior of a set

Definition: Let (X,τ) be a topological space and let A be a subset of X. A point $x \in X$ is said to be an exterior point of A iff it is an interior point of the complement of A, that is, iff there exists an open set G such that $x \in G \subseteq A^c$ or equivalently $x \in G$ and $G \cap A = \phi$. The set all exterior points of A is called the exterior of A and is denoted by A^e or $e(A)$ or $ext(A)$.
i.e $ext(A) = int(A^c)$.

Example: Let $X = \{a,b,c,d\}$ and let $\tau = \{\phi, \{a\}, \{a,c\}, \{a,d\}, \{a,c,d\}, X\}$. Find the exterior points of the set $A = \{b,c\}$.
Solution: $a \in \{a\}, \{a\} \cap \{b,c\} = \phi$.
$\therefore$ a is exterior point of A.
$b \in X, X \cap \{b,c\} = \{b,c\} \neq \phi$.
$\therefore$ b is not exterior point of A.
$c \in \{a,c\}, \{a,c\} \cap \{b,c\} = \{c\} \neq \phi$.
$\therefore$ c is not exterior point of A.
$d \in \{a,d\}, \{a,d\} \cap \{b,c\} = \phi$.
$\therefore$ d is exterior point of A.
$ext(A) = \{a,d\}$.

Remark: (i) $A \cap ext(A) = \phi$.
(ii) $ext(A)$ is open set and is the largest open set contained in A^c.

Theorem: Let (X,τ) be a topological space and $A \subseteq X$. Then $ext(A) = \bigcup\{G \in \tau : G \subseteq A^c\}$.
Proof: By definition, $ext(A) = int(A^c)$ and since $int(A^c) = \bigcup\{G \in \tau : G \subseteq A^c\}$.
Hence $ext(A) = \bigcup\{G \in \tau : G \subseteq A^c\}$.

Remark: (i) $int(A) = ext(A^c) = (cl(A^c))^c$.
(ii) $ext(A) = (cl(A))^c$.

Properties of exterior

Theorem: Let (X,τ) be a topological space and let A, B be any subsets of X. Then:
(i) $ext(X) = \phi$, $ext(\phi) = X$.
(ii) $ext(A) \subseteq A^c$.
(iii) $ext(A) = ext((ext(A))^c)$.
(iv) $A \subseteq B \Longrightarrow ext(B) \subseteq ext(A)$.

(v) $int(A) \subseteq ext\big(ext(A)\big)$.

(vi) $ext(A \cup B) = ext(A) \cap ext(B)$.

Proof: (i) $ext(X) = int(X^c) = int(\phi) = \phi$.

$ext(\phi) = int(\phi^c) = int(X) = X$.

(ii) $ext(A) = int(A^c) \subseteq A^c$ by part (ii) of theorem $[int(A) \subseteq A]$.

(iii) $ext((ext(A))^c) = ext((int(A^c))^c)$

$= int(((int(A^c))^c)^c)$

$= int(int(A^c))$ [since $(A^c)^c = A, int(int(A)) = int(A)$]

$= int(A^c)$

$= ext(A)$.

(iv) $A \subseteq B \Longrightarrow B^c \subseteq A^c \Longrightarrow int(B^c) \subseteq int(A^c)$

$\Longrightarrow ext(B) \subseteq ext(A)$.

(v) By part (ii), we have $ext(A) \subseteq A^c$. Then part (iv) gives $ext(A^c) \subseteq ext(ext(A))$.

But $int(A) = ext(A^c)$. Hence $int(A) \subseteq ext\big(ext(A)\big)$.

(vi) $ext(A \cup B) = int((A \cup B)^c)$

$= int(A^c \cap B^c)$ [By De-Morgan Law]

$= int(A^c) \cap int(B^c)$ By part (iv) of theorem $[int(A \cap B) = int(A) \cap int(B)]$

$= ext(A) \cap ext(B)$.

Frontier of a set

Definition: Let (X, τ) be a topological space and let A be a subset of X. A point $x \in X$ is said to be a frontier point (or boundary point) of A iff it is neither interior nor exterior point of A. The set all frontier points of A is called the frontier of A and is denoted by $Fr_X(A)$ or $Fr(A)$. i.e $Fr(A) = cl(A) \cap (int(A))^c$.

Example: Let $X = \{a, b, c, d\}$ and let $\tau = \{\phi, \{a\}, \{a, c\}, \{a, d\}, \{a, c, d\}, X\}$. Find the frontier points of the set $A = \{b, c\}$.

Solution: $cl(A) = cl(\{b, c\}) = \{b, c\}$, $int(A) = int(\{b, c\}) = \phi$.

$(int(A))^c = \phi^c = X$.

$Fr(A) = cl(A) \cap \big(int(A)\big)^c$

$= \{b, c\} \cap X = \{b, c\}$.

Hence $Fr(A) = \{b, c\}$.

Theorem: Let (X,τ) be a topological space and let A be a subset of X. Then $int(A), ext(A)$ and $Fr(A)$ are disjoint.

Proof: By definition $ext(A) = int(A^c)$. Also $int(A) \subseteq A$ and $int(A^c) \subseteq A^c$.

Since $A \cap A^c = \phi$, it follows that $int(A) \cap ext(A) = int(A) \cap int(A^c) = \phi$.

A gain by the definition of frontier, we have

$$x \in Fr(A) \Leftrightarrow x \notin int(A) \text{ and } x \notin ext(A)$$
$$\Leftrightarrow x \notin int(A) \cup ext(A)$$
$$\Leftrightarrow x \in [int(A) \cup ext(A)]^c.$$

Thus $Fr(A) = [int(A) \cup ext(A)]^c$.

It follows that $Fr(A) \cap int(A) = \phi$ and $Fr(A) \cap ext(A) = \phi$.

Hence $X = int(A) \cup ext(A) \cup Fr(A)$.

Example: Let $X = \{a,b,c,d,e\}$ and let $\tau = \{\phi, \{b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}, X\}$. Find (1) interior (2) exterior (3) frontier of the following subsets of X:

(i) $A = \{c\}$ (ii) $B = \{a,b\}$ (iii) $C = \{a,c,d\}$ (iv) $D = \{b,c,d\}$.

Solution: H.W.

Definition: Let (X,τ) be a topological space and let A, B be subsets of X. Then:

(i) A is said to be dense in B iff $B \subseteq cl(A)$.

(ii) A is said to be dense in X or every where dense iff $cl(A) = X$.

It follows that A is everywhere dense iff every point of X is an adherent point of A.

(iii) A is said to be no where dense or non dense in X iff $int(cl(A)) = \phi$.

(iv) A is said to be dense in itself iff $A \subseteq D(A)$.

Definition: A subset A of a topological space (X,τ) is perfect iff A is dense in itself and closed, that is, iff $A = D(A)$.

Definition: A topological space X is said to be separable iff X contains a countable dense subset, that is, iff there exists a countable subset A of X such that $cl(A) = X$.

Example: The usual topological space $(\mathbb{R}, \mathrm{U})$ is separable.

Solution: Since the set $\mathbb{Q}$ of rational numbers is a countable dense subset of $\mathbb{R}$.

$\mathbb{Q} \in \mathbb{R}$ which is countable and $cl(\mathbb{Q}) = \mathbb{R}$.

Definition: Let (X,τ) be a topological space and let Y be a subset of X we may construct a topology τ_Y for Y which is called the relative topology or the relativization of τ to Y.

Definition: Let (X, τ) be a topological space and let Y be a subset of X the τ-relative topology for Y is the collection τ_Y given by $\tau_Y = \{G \cap Y : G \in \tau\}$.

Remark: The topological space (Y, τ_Y) is called a subspace of (X, τ) the topology τ_Y on Y is called induced by τ.

Example: Consider the topology $\tau = \{\phi, \{1\}, \{3,4\}, \{1,3,4\}, X\}$ on $X = \{1,2,3,4\}$ and the subset $Y = \{1,2,3\}$ of X.

Solution: Let $Y = \{1,2,3\} \subseteq X$. We then have

$\phi \cap \{1,2,3\} = \phi$, $\{1\} \cap \{1,2,3\} = \{1\}$, $\{3,4\} \cap \{1,2,3\} = \{3\}$, $\{1,3,4\} \cap \{1,2,3\} = \{1,3\}$ and $X \cap \{1,2,3\} = \{1,2,3\} = Y$.

Hence the relativization of τ to Y is $\tau_Y = \{\phi, \{1\}, \{3\}, \{1,3\}, Y\}$.

Theorem: Let (X, τ) be a topological space and let Y be a subset of X. Then the collection $\tau_Y = \{G \cap Y : G \in \tau\}$ is a topology on Y.

Proof: H.W.

Definition: A property of a topological space is said to be hereditary if every subspace of the space has that property.

Theorem: Let $(Y, \mathcal{V})$ be a subspace of a topological space (X, τ) and let $(Z, \mathcal{W})$ be a subspace of $(Y, \mathcal{V})$. Then $(Z, \mathcal{W})$ is a subspace of (X, τ).

Theorem: Let (Y, τ_Y) be a subspace of a topological space (X, τ). Then:

(i) a subset A of Y is closed in Y iff there exists a closed set F in X such that $A = F \cap Y$.

(ii) for every $A \subseteq Y$, $cl_Y(A) = cl_X(A) \cap Y$.

(iii) a subset M of Y is a τ_Y-nhd of a point $y \in Y$ iff $M = N \cap Y$ for some τ-nhd N of y.

(iv) a point $y \in Y$ is a τ_Y-limit point of $A \subseteq Y$ iff y is a τ-limit point of A, $D_Y(A) = D(A) \cap Y$.

(v) for every $A \subseteq Y$, $int_Y(A) \supseteq int_X(A)$.

(vi) for every $A \subseteq Y$, $Fr_Y(A) \subseteq Fr_X(A)$.

Proof: (i) A is closed in $Y \Leftrightarrow Y - A$ is open in Y

$\Leftrightarrow$ $Y - A = G \cap Y$ for some open subset G of X

$\Leftrightarrow$ $A = Y - (G \cap Y) = (Y - G) \cup (Y - Y)$ [De-Morgan Law]

$\Leftrightarrow$ $A = Y - G$ [since $Y - Y = \phi$]

$\Leftrightarrow$ $A = Y \cap G^c$

$\Leftrightarrow$ $A = Y \cap F$ where $F = G^c$ is closed in X, since G is open in X.

(ii) By definition, $cl_Y(A) = \bigcap\{K: K \text{ is closed in } Y \text{ and } A \subseteq K\}$

$= \bigcap\{F \cap Y: F \text{ is closed in } X \text{ and } A \subseteq F \cap Y\}$ by (i)

$= [\bigcap\{F: F \text{ is closed in } X \text{ and } A \subseteq F\}] \cap Y$

$= cl_X(A) \cap Y$.

(iii) H.W.

(iv) y is a τ_Y-limit point of $A \Leftrightarrow (M - \{y\}) \cap A \neq \phi, \forall\ \tau_Y$-nhd M of y

$\Leftrightarrow ((N \cap Y) - \{y\}) \cap A \neq \phi, \forall\ \tau$-nhd N of y by (iii)

$\Leftrightarrow (N - \{y\}) \cap A \neq \phi, \forall\ \tau$-nhd N of y

$\Leftrightarrow$ y is a τ-limit point of A.

(v) $x \in int_X(A) \Longrightarrow x$ is a τ-interior point of A

$\Longrightarrow$ A is a τ-nhd of x

$\Longrightarrow$ $A \cap Y$ is a τ_Y-nhd of x by (iii)

$\Longrightarrow$ A is a τ_Y-nhd of x [since $A \subseteq Y \Longrightarrow A \cap Y = A$]

$\Longrightarrow x \in int_Y(A)$.

(vi) H.W.

Example: Give an example to show that in general $int_X(A) \neq int_Y(A)$.

Solution: Let $X = \{a, b, c, d, e\}$ and let $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, e\}, \{a, b, c, d\}, X\}$ be a topological space. Let $Y = \{a, c, e\}$. Then $\tau_Y = \{A \cap Y: A \in \tau\}$ so that the members of τ_Y are: $\phi \cap Y = \phi$, $\{a\} \cap Y = \{a\}$, $\{a, b\} \cap Y = \{a\}$, $\{a, c, d\} \cap Y = \{a, c\}$, $\{a, b, e\} \cap Y = \{a, e\}$, $\{a, b, c, d\} \cap Y = \{a, c\}$ and $X \cap Y = Y$. Thus $\tau_Y = \{\phi, \{a\}, \{a, c\}, \{a, e\}, Y\}$.

Now consider the subset $A = \{a, e\}$ of Y. Then $int_X(A) = \{a\}$ and $int_Y(A) = \{a, e\}$.

Theorem: Let Y be a subspace of a topological space X. If $A \subseteq Y$ is open (closed) in X, then A is also open (closed) in Y.

Proof: H.W.

Theorem: Let (Y, τ_Y) be a subspace of a topological space (X, τ) and let β be a base for τ. Then $\beta_Y = \{B \cap Y: B \in \beta\}$ is a base for τ_Y.

Proof: Let H be a τ_Y-open subset of Y and let $x \in H$. Then there exists a τ-open subset G of X such that $H = G \cap Y$. Since β is a base for τ, there exists a set $B \in \beta$ such that $x \in B \subseteq G$. Since $H \subseteq Y$, it follows that $x \in Y$ and consequently $x \in B \cap Y \subseteq G \cap Y = H$. Thus to each $x \in H$, there exists a member $B \cap Y$ of β_Y such that $x \in B \cap Y \subseteq H$, that is, $H = \bigcup\{B \cap Y: B \cap Y \in \beta_Y \text{ and } B \cap Y \subseteq H\}$. Hence β_Y is a base for τ_Y.