محاضرات مادة بحوث العمليات/المرحلة الرابعه

أستاذ المـادة: م.د. حاجم عات دحام

Operational Research Lectures/ Fourth Stage

Lecturer: Dr. Hajem A. Daham

1 Formulation of LP Models and Graphical Solution

1.1 Introduction to OR

OR problems come in many shapes and sizes - for example:

- What is the best combination of products a manufacturer should produce to maximise profits?

- How should an airline schedule its services?

- When/where should another hydro-electric dam be built? Should it be built at all?

- How should an oil company forecast petrol sales?

- How should electoral boundaries be constructed?

- What combination of facilities will avoid bottlenecks in an emergency clinic?

- What is the most effective layout for a building?

- Under what circumstances does AIDS spread beyond high-risk groups?

- What is the best schedule for delivering a product from a warehouse to customers?

- How much stock of each type should a company hold in inventory?

- How should an investor invest in a share market portfolio to maximise return?

1.2   The Basic OR Process

1. **Definition of the Problem**

   (a) What are the **goals** or **objectives**?

   (b) What are the **decision alternatives** of the system?

   (c) What are the **limitations**, **restrictions** and **requirements** of the system?

2. **Model Formulation**

   Formulate a mathematical model of the problem, with **decision variables** and **objective function** and **constraints**.

3. **Solution of the Model**

   (a) For 'clean', 'clinical' problems: find a solution by <u>analytical</u> or <u>computational</u> methods.

   (b) For 'complex' problems: use <u>simulation</u>.

   (c) After solving: apply **sensitivity analysis** to test the robustness and behaviour of the system to changes in the parameters.

4. **Model Validation**

   Can we reproduce the actual performance of the system?

5. **Implementation**

   Put the recommendations of the analysis into practice.

> **Reading**: Winston Chapter 1

> **Reading**: Revise Winston Chapter 2 on Linear Algebra, especially **row operations**.

1.3   An Example LP Problem

The MachineCo Problem

MachineCo produces earthmoving equipment (E) and forestry equipment (F).

> Profit on one unit of E: $5000
> Profit on one unit of F: $4000

Production of each unit requires machining operations in departments A and B.

> **Department A**: 150 hours available
> **Department B**: 160 hours available

> **E machining**: 10 hours in dep't A and 20 hours in dep't B
> **F machining**: 15 hours in dep't A and 10 hours in dep't B

Union regulations require that labour hours used in the testing of finished products (undertaken in department C, unrelated to A and B) cannot be less than 135 hours.

> **E testing**: 30 hours.
> **F testing**: 10 hours.

In order to maintain market position, management operating policy is:

> build at least one F for every 3 Es.

A major customer has ordered a total of 5 pieces of equipment (in any combination). MachineCo's goal is therefore:

> determine the number of Es and the number of Fs to produce in order to **maximise the overall profit**, while complying with all the restrictions.

1. **The Decision Variables**

   The <u>Decision Variables</u> are the quantities *that we can control* in order to achieve the stated goal (e.g. "maximum profit"). In the current situation, the quantities that are under MachineCo's control are the numbers of E and F that should be produced (and sold to the customer).

2. **The Objective Function**

   The <u>Objective Function</u> is the quantity we wish to achieve the "best value" for (i.e. to *optimise*). For MachineCo's problem, that quantity is their **profit**. Since they make a profit of $5000 for <u>each</u> earthmover and $4000 for <u>each</u> forestry machine, their **total profit** will be

$$\$(5000E + 4000F).$$

3. **The Constraints**

   The <u>constraints</u> are the *restrictions* that impose limits on the values that the decision variables can take. In MachineCo's case, there are a number of these limitations:

   (a) *Available hours in Department A*

   (b) *Available hours in Department B*

   (c) *Available testing hours in Department C*

   (d) *Production mix*

   (e) *Minimum production*

   (f) *Sign restrictions*

(a) *Available hours in Department A*

Since <u>each</u> earthmover needs 10 hours in Department A and <u>each</u> forestry machine needs 15 hours, the total time in Department A is $10E + 15F$ and this must not be greater than the available time of 150 hours. Hence, we have:

$$\boxed{10E + 15F \leq 150.}$$

(b) *Available hours in Department B*

A similar argument for Department B leads to

$$\boxed{20E + 10F \leq 160.}$$

(c) *Available testing hours in Department C*

A similar argument for Department C leads to (note the **direction** of the inequality in this case):

$$\boxed{30E + 10F \geq 135.}$$

(d) *Production mix*

We are required to build at least one F for every three Es. Mathematically, we can write this as $F \geq \frac{1}{3}E$, but (for reasons that will become clear later) we prefer to write it as

$$\boxed{E - 3F \leq 0.}$$

(e) *Minimum production*

In order to fulfil the order, we need to produce at least five machines in total:

$$\boxed{E + F \geq 5.}$$

(f) *Sign restrictions*

We **cannot** produce <u>negative</u> numbers of machines, so we must include the pair of constraints

$$\boxed{E, F \geq 0.} \qquad \leftarrow \textbf{VITAL!}$$

So we can now express the whole problem <u>mathematically</u> as follows:

Full Model for the MachineCo Problem

$$\max \quad z = 5000E + 4000F$$

subject to (s.t.)

$$
\begin{aligned}
10E + 15F &\leq 150 \\
20E + 10F &\leq 160 \\
30E + 10F &\geq 135 \\
E - 3F &\leq 0 \\
E + F &\geq 5 \\
E, F &\geq 0 \, .
\end{aligned}
$$

UNIT 2 LINEAR PROGRAMMING

Structure

2.0 INTRODUCTION

Linear programming deals with the optimization (maximization or minimization) of a function of variables known as objective functions. It is subject to a set of linear equalities and/or inequalities known as *constraints*. Linear programming is a mathematical technique which involves the allocation of limited resources in an optimal manner on the basis of a given criterion of optimality.

In this unit, the properties of Linear Programming Problems (LPP) are discussed. The graphical method of solving a LPP is applicable where two variables are involved. The most widely used method for solving LP problems consisting of any number of variables is called the *simplex method* developed by G. Dantzig in 1947 and made generally available in 1951.

2.1 UNIT OBJECTIVES

After going through this unit, you will be able to:
- Describe the procedure for mathematical formulation of LPP
- Explain the procedure for solving LPP by graphical method
- Describe the steps involved in solving LPP using the Big M method

- Understand the procedure of solving LPP using the simplex method
- Understand the concept of duality in LP

2.2 LINEAR PROGRAMMING FORMULATION

The procedure for mathematical formulation of an LP problem consists of the following steps:

Step 1 Write down the decision variables of the problem.

Step 2 Formulate the objective function to be optimized (Maximized or Minimized) as a linear function of the decision variables.

Step 3 Formulate the other conditions of the problem such as resource limitation, market constraints and interrelations between variables as linear inequations or equations in terms of the decision variables.

Step 4 Add the non-negativity constraint from the considerations so that the negative values of the decision variables do not have any valid physical interpretation.

The objective function, the set of constraint and the non-negative constraint together form a linear programming problem.

2.2.1 General Formulation of LPP

The general formulation of the LPP can be stated as follows:

In order to find the values of n decision variables $x_1 x_2 \ldots x_n$ maximize or minimize the objective function.

$$Z \quad c_1 x_1 \quad c_2 x_2 \quad \ldots \quad c_n x_n \qquad \ldots(1)$$

$$
\begin{aligned}
&a_{11} x_1 \quad a_{12} x_2 \quad \cdots \quad a_{1n} x_n (\quad) b_1 \\
&a_{21} x_1 \quad a_{22} x_2 \quad \cdots \quad a_{2n} x_n (\quad) b_2 \\
&\vdots \\
&a_{i1} x_1 \quad a_{i2} x_2 \quad \cdots \quad a_{in} x_n (\quad) b_i \\
&\vdots \\
&a_{m1} x_1 \quad a_{m2} x_2 \quad \cdots \quad a_{mn} x_n (\quad) b_n
\end{aligned}
\qquad \ldots(2)
$$

where constraints may be in the form of inequality $\leq$ or $\geq$ or even in the form an equation (=) and finally satisfy the non negative restrictions

$$x_1 \quad 0, x_2 \quad 0 \ldots x_n \quad 0 \qquad \ldots(3)$$

2.2.2 Matrix Form of LPP

The LPP can be expressed in the matrix form as follows:

Maximize or

Minimize $Z = Cx \rightarrow$ Objective function

subject to $Ax \ (\leq = \geq) \ b$ Constant equation

$b > 0, x \geq 0$ Non-negativity restrictions,

where $\quad x = (x_1 \ x_2 \cdots x_n)$

$\qquad c = (c_1 \ c_2 \cdots c_n)$

$$b = \begin{matrix} b_1 \\ b_2 \\ b_m \end{matrix} \quad A \quad \begin{matrix} a_{11}a_{12}\cdots a_{1n} \\ a_{21}a_{22}\cdots a_{2n} \\ : \\ a_{m1}a_{m2}\cdots a_{mn} \end{matrix}$$

Example 2.1: *A manufacturer produces two types of models M_1 and M_2. Each model of the type M_1 requires 4 hours of grinding and 2 hours of polishing; whereas each model of the type M_2 requires 2 hours of grinding and 5 hours of polishing. The manufacturers have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works for 60 hours a week. Profit on M_1 model is Rs 3.00 and on model M_2 is Rs 4.00. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models, so that he may make the maximum profit in a week?*

Solution:

Decision variables: *Let X_1 and X_2 be the number of units of M_1 and M_2 model.*

Objective function: *Since the profit on both the models are given, we have to maximize the profit, viz.*

Max $Z = 3X_1 + 4X_2$

Constraints: *There are two constraints, one for grinding and the other for polishing.*

The number of hours available on each grinder for one week is 40 hrs. There are two grinders. Hence, the manufacturer does not have more than $2 \times 40 = 80$ hours of grinding. M_1 requires 4 hours of grinding and M_2 requires 2 hours of grinding.

The grinding constraint is given by

$4x_1 \quad 2x_2 \quad 80$

Since there are 3 polishers, the available time for polishing in a week is given by $3 \times 60 = 180$. M_1 requires 2 hours of polishing and M_2 requires 5 hours of polishing. Hence, we have $2x_1 + 5x_2 \leq 180$.

Finally we have

Max $Z = 3x_1 + 4x_2$

Subject to $4x_1 \quad 2x_2 \quad 80$

$2x_1 \quad 5x_2 \quad 180$

$x_1 x_2 \quad 0$

Example 2.2: *A company manufactures two products A and B. These products are processed in the same machine. It takes 10 minutes to process one unit of product A and 2 minutes for each unit of product B and the machine operates for a maximum of 35 hours in a week. Product A requires 1 kg. and B 0.5 kg. of raw material per unit the supply of which is 600 kg. per week. Market constraint on product B is known to be 800 units every week. Product A costs Rs 5 per unit and sold at Rs 10. Product B costs Rs 6 per unit and can be sold in the market at a unit price of Rs 8. Determine the number of units of A and B per week to maximize the profit.*

Solution:

Decision variables: Let x_1 and x_2 be the number of products A and B.

Objective function: Costs of product A per unit is Rs 5 and sold at Rs 10 per unit.

$\therefore$ Profit on one unit of product A

$= 10 - 5 = 5$

$\therefore$ x_1 units of product A contributes a profit of Rs 5 profit contribution from one unit of product

$B = 8 - 6 = 2$

$\therefore$ x_2 units of product B contribute a profit of Rs 2

$\therefore$ The objective function is given by

Max Z $5x_1$ $2x_2$

Constraints: Time requirement constraint is given by

$10x_1$ $2x_2$ $(35$ $60)$
$10x_1$ $2x_2$ 21.00

Raw material constraint is given by

x_1 $0.5x_2$ 600

Market demand on product B is 800 units every week.

$\therefore$ $x_2 \geq 800$

The complete LPP is

Max Z $5x_1$ $2x_2$

Subject to $10x_1$ $2x_2$ 2100

x_1 $0.5x_2$ P 600

x_2 800

x_1, x_2 0

Example 2.3: A person requires 10,12, and 12 units of chemicals A, B respectively for his garden. A liquid product contains 5,2 and 1 units of A, B and C respectively per jar. A dry product contains 1,2 and 4 units of A, B, C per carton. If the liquid product sells for Rs 3 per jar and the dry product sells for Rs 2 per carton, how many of each should be purchased, in order to minimize the cost and meet the requirements?

Solution:

Decision variables: Let X_1 and X_2 be the number of units of liquid and dry products.

Objective function: Since the cost for the products are given we have to minimize the cost.

Min $Z = 3x_1 + 2x_2$

Constraints: As there are three chemicals and its requirement are given, we have three constraints for these three chemicals.

$$5x_1 \quad x_2 \quad 10$$
$$2x_1 \quad 2x_2 \quad 12$$
$$x_1 \quad 4x_2 \quad 12$$

Finally, the complete LPP is

Min $Z = 3x_1 + 2x_2$

Subject to

$$5x_1 \quad x_2 \quad 10$$
$$2x_1 \quad 2x_2 \quad 12$$
$$x_1 \quad 4x_2 \quad 12$$
$$x_1, x_2 \quad 0$$

Example 2.4: A paper mill produces two grades of paper, namely *X* and Y. Because of raw material restrictions, it cannot produce more than 400 tonnes of grade *X* and 300 tonnes of grade *Y* in a week. There are 160 production hours in a week. It requires 0.2 and 0.4 hours to produce a tonne of products *X* and *Y* respectively with corresponding profits of Rs 200 and Rs 500 per tonne. Formulate this as a LPP to maximize profit and find the optimum product mix.

Solution:

Decision variables: Let x_1 and x_2 be the number of units of two grades of paper *X* and *Y*.

Objective function: Since the profit for the two grades of paper *X* and *Y* are given, the objective function is to maximize the profit.

Max $Z = 200x_1 + 500x_2$

Constraints: There are 2 constraints one w.r.t. to raw material, and the other w.r to concerning production hours.

Max $Z = 200x_1 + 500x_2$

Subject to

$$x_1 \quad 400$$
$$x_2 \quad 300$$
$$0.2x_1 \quad 0.4x_2 \quad 160$$

Non-negative restriction $x_1 \ x_2 \geq 0$

Example 2.5: A company manufactures two products *A* and *B*. Each unit of *B* takes twice as long to produce as one unit of *A* and if the company were to produce only *A* it would have time to produce 2000 units per day. The availability of the raw material is sufficient to produce 1500 units per day of both *A* and *B* combined. Product *B* requiring a special ingredient only 600 units can be made per day. If *A* fetches a profit of Rs 2 per unit and *B* a profit of Rs 4 per unit, find the optimum product mix by graphical method.

Solution: Let x_1 and x_2 be the number of units of the products *A* and *B* respectively.

The profit after selling these two products is given by the objective function

Max $Z = 2x_1 + 4x_2$

Since the company can produce at the most 2000 units of the product in a day and type *B* requires twice as much time as that of type *A*, production restriction is given by

$$x_1 \quad 2x_2 \quad 2000$$

Since the raw materials are sufficient to produce 1500 units per day both *A* and *B* combined we have $x_1 \quad x_2 \quad 1500$.

There are special ingredients for the product *B*; we have $x_2 \le 600$.

Also, since the company cannot produce negative quantities $x_1 \ge 0$ and $x_2 \ge 0$.

Hence, the problem can be finally put in the form:

Find x_1 and x_2 such that the profits

$Z = 2x_1 + 4x_2$ is maximum.

$$x_1 \quad 2x_2 P \quad 2000$$
$$x_1 \quad x_2 \quad 1500$$
Subject to $\qquad x_2 \quad 600$
$$x_1, x_2 \quad 0$$

Example 2.6: *A* firm manufacturers 3 products *A*, *B* and *C*. The profits are Rs 3, Rs 2 and Rs 4 respectively. The firm has 2 machines and the followign is the required processing time in minutes for each machine on each product.

		Product		
		A	*B*	*C*
Machines	*C*	4	3	5
	D	3	2	4

Machine *C* and *D* have 2000 and 2500 machine minutes respectively. The firm must manufacture 100 units of *A*, 200 units of *B* and 50 units of *C* but no more than 150 *A's*. Set up an LP problem to maximize the profit.

Solution: Let x_1, x_2, x_3 be the number of units of the product *A, B, C* respectively.

Since the profits are Rs 3, Rs 2 and Rs 4 respectively, the total profit gained by the firm after selling these three products is given by

$$Z \quad 3x_1 \quad 2x_2 \quad 4x_3$$

The total number of minutes required in producing these three products at machine *C* is given by $4x_1 + 3x_2 + 5x_3$ and at machine *D* is given by $2x_1 + 2x_2 + 4x_3$.

The restrictions on the machine *C* and *D* are given by 2000 minutes and 2500 minutes.

$$4x_1 \quad 3x_2 \quad 5x_3 \quad 2000$$
$$2x_1 \quad 2x_2 \quad 4x_3 \quad 2500$$

Also, since the firm manufactures 100 units of *A*, 200 units of *B* and 50 units of *C* but not more than 150 unit of *A* the further restriction becomes

2.3.1 Procedure for Solving LPP by Graphical Method

The steps involved in the graphical method are as follows.

Step 1   Consider each inequality constraint as an equation.

Step 2 Plot each equation on the graph as each will geometrically represent a straight line.

Step 3 Mark the region. If the inequality constraint corresponding to that line is O then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality constraint P sign, the region above the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. The common region, thus obtained, is called the feasible region.

Step 4   Assign an arbitrary value, say zero, for the objective function.

Step 5   Draw a straight line to represent the objective function with the arbitrary value (i.e., a straight line through the origin).

Step 6 Stretch the objective function line till the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin, passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passes through at least one corner of the feasible region.

Step 7   Find the coordinates of the extreme points selected in step 6 and find the maximum or minimum value of Z.

Note: As the optimal values occur at the corner points of the feasible region, it is enough to calculate the value of the objective function of the corner points of the feasible region and select the one which gives the optimal solution, i.e., in the case of maximization problem the optimal point corresponds to the corner point at which the objective function has a maximum value and in the case of minimization, the corner point which gives the objective function the minimum value is the optimal solution.

Example 2.9: Solve the following LPP by graphical method.

Minimize $Z = 20X_1 + 10X_2$

Subject ot $X_1 + 2X_2 \leq 40$

$3X_1 + X_2 \geq 30$

$4X_1 + 3X_2 \geq 60$

$X_1, X_2 \geq 0$

Solution: Replace all the inequalities of the constraints by equation

$X_1 + 2X_2 = 40$ If $X_1 = 0 \Rightarrow X_2 = 20$

                 If $X_2 = 0 \Rightarrow X_1 = 40$

$\therefore X_1 + 2X_2 = 40$ passes through (0, 20) (40, 0)

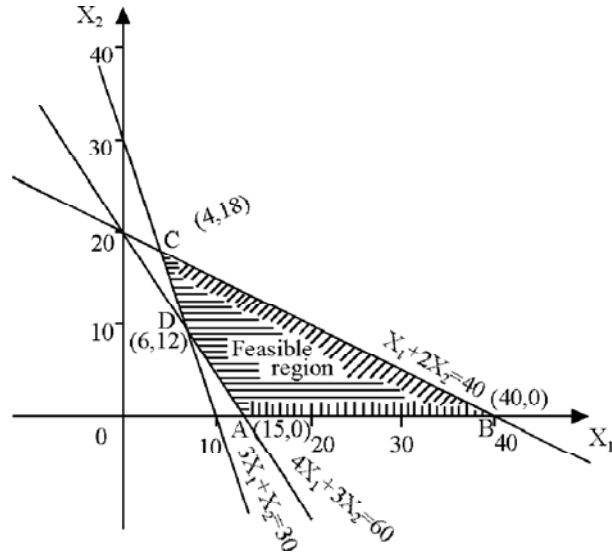$3X_1 + X_2 = 30$ passes through (0, 30) (10, 0)

$4X_1 + 3X_2 = 60$ passes through (0, 20) (15, 0)

Plot each equation on the graph.

The feasible region is *ABCD*.

C and *D* are the points of intersection of lines

$X_1 + 2X_2 = 40$, $3X_1 + X_2 = 30$ and

$4X_1 + 3X_2 = 60$, $X_1 + X_2 = 30$

On solving we get $C = (4, 18)$ $D = (6, 12)$

Corner points	Value of $Z = 20X_1 + 10X_2$
A (15, 0)	300
B (40, 0)	800
C (4, 18)	260
D (6, 12)	240 (Minimum value)

∴ The minimum value of *Z* occurs at *D* (6, 12). Hence, the optimal solution is $X_1 = 6$, $X_2 = 12$.

Example 2.10: Find the maximum value of $Z = 5X_1 + 7X_2$.

Subject to the constraints,

$X_1 + X_2 \leq 4$

$3X_1 + 8X_2 \leq 24$

$10X_1 + 7X_2 \leq 35$

$X_1, X_2 > 0$

Solution: Replace all the inequalities of the constraints by forming equations.
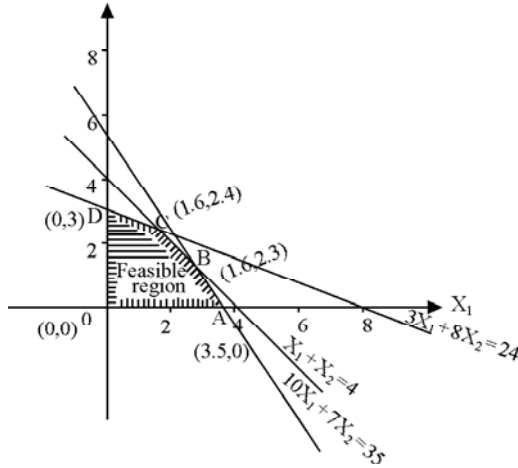
$X_1 + X_2 = 4$ passes through (0, 4) (4, 0)

$3X_1 + 8X_2 = 24$ passes through (0, 3) (8, 0)

$10X_1 + 7X_2 = 35$ passes through (0, 5) (3.5, 0)

Plot these lines in the graph and mark the region below the line since the inequality of the constraint is ≤ and is also lying in the first quadrant.

The feasible region is *OABCD*.

B and C are points of intersection of lines.

$X_1 + X_2 = 4$, $10X_1 + 7X_2 = 35$ and

$3X_1 + 8X_2 = 24$, $X_1 + X_2 = 4$

On solving we get,

B = (1.6, 2.3)

C = (1.6, 2.4)

Corner points	*Value of Z* $= 5X_1 + 7X_2$
O (0, 0)	0
A (3.5, 0)	17.5
B (1.6, 2.3)	25.1
C (1.6, 2.4)	24.8 (Maximum value)
D (0, 3)	21

∴ The maximum value of Z occurs at *C* (1.6, 2.4) and the optimal solution is $X_1 = 1.6$, $X_2 = 2.4$.

Example 2.11: A company produces 2 types of hats. Every hat of type *A* requires twice as much labour time than the second type B. If the company proclues only hat *B* then it can produce a total of 500 hats a day. The market limits daily sales of the hat *A* and hat *B* to 150 and 250 hats. The profits on hat *A* and *B* are Rs 8 and Rs 5 respectively. Solve graphically to get the optimal solution.

Solution: Let X_1 and X_2 be the number of units of type *A* and type *B* hats respectively.

Max $Z = 8X_1 + 5X_2$

Subject to $\quad\quad 2X_1 + 2X_2 \leq 500$

$X_1 \geq 150$

$X_2 \geq 250$

$X_1, X_2 \geq 0$.

First rewrite the inequality of the constraint into an equation and plot
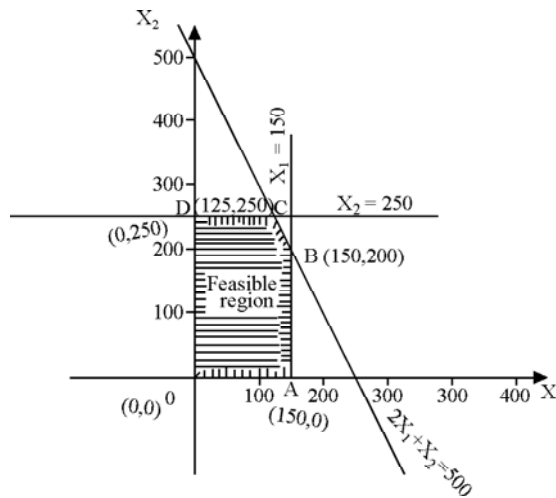
the lines in the graph.

$2X_1 + X_2 = 500$ passes through (0, 500) (250, 0)

$X_1 = 150$ passes through (150, 0)

$X_2 = 250$ passes through (0, 250)

We mark the region below the lines lying in the first quadrant since the inequality of the constraints is $\leq$. The feasible region is *OABCD* and *B* and *C* are points of intersection of lines

$2X_1 + X_2 = 500$, $X_1 = 150$ and

$2X_1 + X_2 = 500$, $X_2 = 250$.

On solving, we get $B = (150, 200)$

$$C = (125, 250)$$

Corner points	Value of $Z = 8X_1 + 5X_2$
O (0,0)	0
A (150,0)	1200
B (150,200)	2200
C (125,250)	2250 (Maximum Z = 2250)
D (0,250)	1250

The maximum value of *Z* is attained at *C* (125, 250).

$\therefore$ The optimal solution is $X_1 = 125$, $X_2 = 250$.

i.e., the company should produce 125 hats of type A and 250 hats of type

B in order to get the maximum profit of Rs 2250.
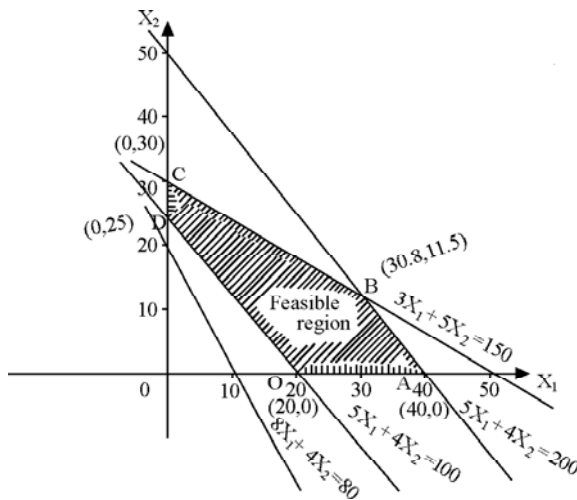
Example 2.12: By graphical method, solve the following LPP.

Max Z = $3X_1 + 4X_2$

Subject to $5X_1 + 4X_2 \leq 200$

                $3X_1 + 5X_2 \leq 150$

                $5X_1 + 4X_2 \geq 100$

                $8X_1 + 4X_2 \geq 80$

and $X_1, X_2 \geq 0$

Solution:

Feasible region is given by *OABCD*.

Corner points	Value of [Z 3X₁] 4X₂	
O (20, 0)	60	
A (40, 0)	120	
B (30.8, 11.5)	138.4	(Maximum value)
C (0, 30)	120	
D (0, 25)	100	

$\therefore$ The maximum value of Z is attained at *B* (30.8,11.5).

$\therefore$ The optimal solution is $X_1 = 30.8$, $X_2 = 11.5$.

Example 2.13: Use graphical method to solve the LPP.
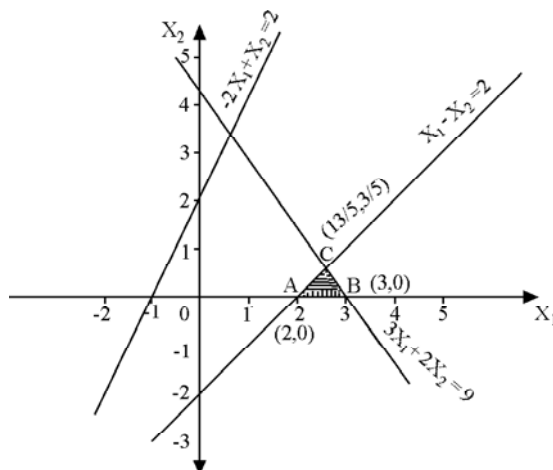
Maximize $Z = 6X_1 + 4X_2$

Subject to $\qquad -2X_1 + X_2 \le 2$

$X_1 - X_2 \le 2$

$3X_1 + 2X_2 \le 9$

$X_1, X_2 \ge 0$

Solution:

Feasible region is given by *ABC*.

Corner points	Value of $Z = 3X_1 + 4X_2$
A (2, 0)	12
B (3,0)	18
C (13/5, 3/5)	98/5 = 19.6 (Maximum value)

The maximum value of Z is attained at c (13/5, 3/5).

∴ The optimal solution is $X_1 = 13/5$, $X_2 = 3/5$.

Example 2.14: Use graphical method to solve the LPP.

Maximize $Z = 3X_1 + 2X_2$

Subject to
$$5X_1 + X_2 \geq 10$$
$$X_1 + X_2 \geq 6$$
$$X_1 + 4X_2 \geq 12$$
$$X_1, X_2 \geq 0.$$

Solution:

Corner points	Value of $Z = 3X_1 + 2X_2$	
A (0, 10)	20	
B (1,5)	13	(Minimum value)
C (4, 2)	16	
D (12, 0)	36	
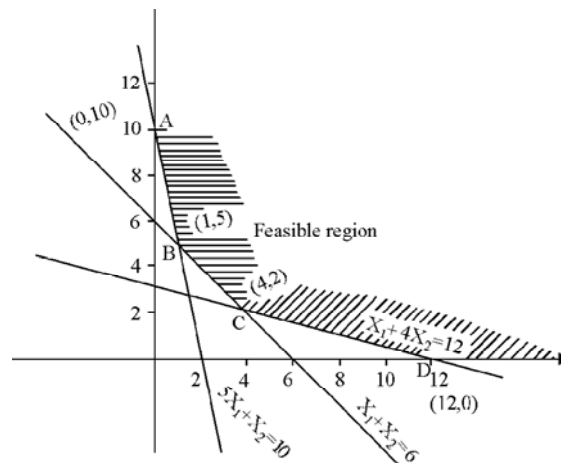
Since the minimum value is attained at *B* (1,5) the optimum solution is X_1, $X_2 = 5$.

Note: In this problem if the objective function is maximization then the solution is unbounded, as the maximum value of Z occurs at infinity.

Some more cases

There are some linear programming problems which may have

(i) A unique optimal solution

(ii) An infinite number of optimal solution

Simplex method

Simplex method is the method to solve (LPP) models which contain two or more decision variables.

Basic variables:

Are the variables which coefficients <u>One</u> in the equations and <u>Zero</u> in the other equations.

Non-Basic variables:

Are the variables which coefficients are taking any of the values, whether positive or negative or zero.

Slack, surplus & artificial variables:

a) If the inequality be $\leq$ (less than or equal, then we add a slack variable + S to change $\leq$ to =.

b) If the inequality be $\geq$ (greater than or equal, then we subtract a surplus variable - S to change $\geq$ to =.

c) If we have = we use artificial variables.

The steps of the simplex method:

Step 1:
Determine a starting basic feasible solution.

Step 2:
Select an entering variable using the optimality condition. Stop if there is no entering variable.

Step 3:
Select a leaving variable using the feasibility condition.

Optimality condition:

The entering variable in a maximization (minimization) problem is the non-basic variable having the most negative (positive) coefficient in the Z-row.

The optimum is reached at the iteration where all the Z-row coefficient of the non-basic variables are non-negative (non-positive).

Feasibility condition:

For both maximization and minimization problems the leaving variable is the basic associated with the smallest non-negative ratio (with strictly positive denominator).

Pivot row:

a) Replace the leaving variable in the basic column with the entering variable.

b) New pivot row equal to current pivot row divided by pivot element.

c) All other rows:

New row=current row - (pivot column coefficient) *new pivot row.

Example 1:

Use the simplex method to solve the (LP) model:

$$max \ Z = 5x_1 + 4x_2$$

Subject to

$$6x_1 + 4x_2 \leq 24$$
$$x_1 + 2x_2 \leq 6$$
$$-x_1 + x_2 \leq 1$$
$$x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Solution:

$$max \ Z - 5x_1 + 4x_2 = 0$$

Subject to

$$6x_1 + 4x_2 + S_1 = 24$$
$$x_1 + 2x_2 + S_2 = 6$$
$$-x_1 + x_2 + S_3 = 1$$
$$x_2 + S_4 = 2$$

Table 1:

Basic	x_1	x_2	S_1	S_2	S_3	S_4	Sol.
S_1	6	4	1	0	0	0	24
S_2	1	2	0	1	0	0	6
S_3	-1	1	0	0	1	0	1
S_4	0	1	0	0	0	1	2
Max Z	-5	-4	0	0	0	0	0

$$\frac{24}{6} = 4$$

$$\frac{6}{1} = 6$$

$$\frac{1}{-1} = -1 \quad \text{(ignore)}$$

$$\frac{2}{0} = \infty \quad \text{(ignore)}$$

The entering variable is x_1 and S_1 is a leaving variable.

Table 2:

Basic	x_1	x_2	S_1	S_2	S_3	S_4	Sol.
x_1	1	2/3	1/6	0	0	0	4
S_2	0	4/3	-1/6	1	0	0	2
S_3	0	5/3	1/6	0	1	0	5
S_4	0	1	0	0	0	1	2
Max Z	0	-2/3	5/6	0	0	0	20

■ **Pivot row or new x_1-row=$\frac{1}{6}$ [current S_1 −row]**

$$= \frac{1}{6}[6 \quad 4 \quad 1 \quad 0 \quad 0 \quad 0 \quad 24]$$

$$= [1 \quad \frac{2}{3} \quad \frac{1}{6} \quad 0 \quad 0 \quad 0 \quad 4]$$

- New S_2-row=[current S_2 −row]-(1)[new x_1 −row]

=[1 2 0 1 0 0 6]- (1)[1 2/3 1/6 0 0 0 0 4]

=[0 4/3 -1/6 1 0 0 2]

- New S_3-row=[current S_3 −row]-(1)[new x_1 −row]

=[-1 1 0 0 1 0 1]- (1)[1 2/3 1/6 0 0 0 0 4]

=[0 5/3 1/6 0 1 0 5]

- New S_4-row=[current S_4 −row]-(0)[new x_1 −row]

=[0 1 0 0 0 1 2]- (0)[1 2/3 1/6 0 0 0 0 4]

=[0 1 0 0 0 1 2]

- New Z-row=[current Z −row]-(-5)[new x_1 −row]

=[-5 -4 0 0 0 0 0]-(-5)[1 2/3 1/6 0 0 0 0 4]

=[0 -2/3 5/6 0 0 0 20]

Now:

$$\frac{4}{\frac{2}{3}} = 6$$

$$\frac{2}{\frac{4}{3}} = \frac{6}{4} = \boxed{\frac{3}{2}}$$

$$\frac{5}{\frac{5}{3}} = 3$$

$$\frac{2}{1} = 2$$

The entering variable is x_2 and S_2 is a leaving variable.

Table 3: (optimal solution):

Basic	x_1	x_2	S_1	S_2	S_3	S_4	Sol.
x_1	1	0	1/4	-1/2	0	0	3
x_2	0	1	-1/8	3/4	0	0	3/2
S_3	0	0	3/8	-5/4	1	0	5/2
S_4	0	0	1/8	-3/4	0	1	1/2
Max Z	0	0	5/6	1/2	0	0	21

∎ **Pivot row or new x_2-row=$\dfrac{1}{\frac{4}{3}}$ [current S_2 –row]**

$=\dfrac{1}{\frac{4}{3}}[0\quad 4/3\quad -1/6\quad 1\quad 0\quad 0\quad 2]$

$=[0\quad 1\quad -1/8\quad ¾\quad 0\quad 0\quad 3/2]$

- **New x_1-row=[current x_1 –row]-(2/3)[new x_2 –row]**

$=[1\quad 2/3\quad 1/6\quad 0\quad 0\quad 0\quad 4]- (2/3)[0\quad 1\quad -1/8\quad ¾\quad 0\quad 0\quad 3/2]$

$=[1\quad 0\quad ¼\quad -1/2\quad 0\quad 0\quad 3]$

- **New S_3-row=[current S_3 –row]-(5/2)[new x_2 –row]**

$=[0\quad 5/3\quad 1/6\quad 0\quad 1\quad 0\quad 5]-(5/3)[0\quad 1\quad -1/8\quad ¾\quad 0\quad 0\quad 3/2]$

$=[0\quad 0\quad 3/8\quad -5/4\quad 1\quad 0\quad 5/3]$

- **New S_4-row=[current S_4 –row]-(1)[new x_2 –row]**

$=[0\quad 1\quad 0\quad 0\quad 0\quad 1\quad 2]-(1)[0\quad 1\quad -1/8\quad ¾\quad 0\quad 0\quad 3/2]$

$=[0\quad 0\quad 1/8\quad -3/4\quad 0\quad 1\quad ½]$

New Z-row=[current Z –row]-(-2/3)[new x_2 –row]

$=[0\quad -2/3\quad 5/6\quad 0\quad 0\quad 0\quad 20]-(-2/3)[0\quad 1\quad -1/8\quad ¾\quad 0\quad 0\ 3/2]$

$=[0\quad 0\quad ¾\quad ½\quad 0\quad 0\quad 21]$

Then the solution is:

$$x_1 = 3 \ \& \ x_2 = \frac{3}{2} \ \& \ S_3 = \frac{5}{2} \ \& \ S_4 = \frac{1}{2}$$

$$S_1 = 0, \ S_2 = 0$$

Example 2:

Use the simplex method to solve the (LP) model:

$$max \ Z = 2x_1 + 3x_2$$

Subject to

$$0.25x_1 + 0.5x_2 \leq 40$$

$$0.4x_1 + 0.2x_2 \ \leq 40$$

$$0.8x_2 \ \ \leq 40$$

$$x_1, x_2 \ \ \geq 0$$

Solution:

$$max \ Z - 2x_1 + 3x_2 = 0$$

Subject to

$$0.25x_1 + 0.5x_2 + S_1 = 40$$

$$0.4x_1 + 0.2x_2 + S_2 \ = 40$$

$$0.8x_2 + S_3 \ = 40$$

$$x_1, x_2, +S_1, +S_2, +S_3 \ \geq 0$$

Table 1:

Basic	x_1	x_2	S_1	S_2	S_3	Sol.
S_1	0.25	0.5	1	0	0	40
S_2	0.4	0.2	0	1	0	40
S_3	0	0.8	0	0	1	40
Max Z	-2	-3	0	0	0	0

$$\frac{40}{0.5} = 80$$

$$\frac{40}{0.2} = 200$$

$$\frac{40}{0.8} = \boxed{50}$$

- ■ **Pivot row or new S_3-row**$=\frac{1}{0.8}$[0   0.8   0   0   1   40]

  =[0   1   0   0   1.25   50]

New S_1-row=[current S_1 –row]-(0.5)[new x_2 –row]

=[0.25   0.5   1   0   0   40]-(0.5)[0   1   0   0   1.25   50]

=[0   0.5   0   0   -0.625   15]

New S_2-row=[current S_2 –row]-(0.2)[new x_2 –row]

=[0.4   0.2   0   1   0   40]-(0.2)[0   1   0   0   1.25   50]

[0.4   0   0   1   -0.25   30]

New Z-row=[current Z –row]-(-3)[new x_2 –row]

=[-2   -3   0   0   0   0]-(-3)[0   1   0   0   1.25   50]

=[-2   0   0   0   3.75   150]

Table 2:

Basic	x_1	x_2	S_1	S_2	S_3	Sol.
S_1	0.25	0	1	0	-0.625	15
S_2	0.4	0	0	1	-0.25	30
x_2	0	1	0	0	1.25	50
Max Z	-2	0	0	0	3.75	150

$$\frac{15}{0.25} = 60$$

$$\frac{30}{0.4} = 75$$

$$\frac{50}{0} = \infty \quad \text{(ignore)}$$

Pivot row or new S_1-row=$\frac{1}{0.25}$ [0.25 0 1 0 -0.625 15]

=[1 0 4 0 -2.5 60]

New S_2-row=[current S_2 –row]-(0.4)[new x_1 –row]

=[0.4 0 0 0 -0.25 30]-(0.4)[1 0 4 0 -2. 5 60]

[0 0 -1.6 0 -0.75 6]

New x_2-row=[0 1 0 0 1.25 50]-(0)[1 0 4 0 -2. 5 60]

=[0 1 0 0 1.25 50]

New Z-row=[current Z –row]-(-2)[1 0 4 0 -2.5 60]

=[-2 0 0 0 3.75 150]-(-2)[1 0 4 0 -2. 5 60]

[0 0 8 0 -1.25 270]

Table 3:

Basic	x_1	x_2	S_1	S_2	S_3	Sol.
x_1	1	0	4	0	-2.5	60
S_2	0	0	-1.6	1	0.75	6
x_2	0	1	0	0	1.25	50
Max Z	0	0	8	0	-1.25	270

$$\frac{60}{-2.5} = -24 \text{ (ignore)}$$

$$\frac{6}{0.75} = 8$$

$$\frac{50}{1.25} = 40$$

New S_2-row=$\frac{1}{0.75}$ =[current S_2-row] =$\frac{1}{0.75}$ [0 0 -1.6 0 0.75   6]

$\qquad$ =[0   0 -2.133   0 1   8]

New x_1-row= [1 0 4 0 -2.5   60]-(-2.5)[0   0 -2.133   0 1   8]

$\qquad$ =[1   0 -1.333   0 0   80]

New x_2-row= [0 1 0 0 1.25   50]-(-1.25)[0   0 -2.133   0 1   8]

$\qquad$ =[0   1 -2.76   0 0   40]

New Z-row= [0 0 8 0 -1.25   270]-(-2.5)[0 0 -2.133   0 1   8]

$\qquad$ =[0   0 5.33   0 0   280]

Table 3: (optimal solution):

Basic	x_1	x_2	S_1	S_2	S_3	Sol.
x_1	1	0	-1.333	0	0	80
S_3	0	0	-2.133	0	1	8
x_2	0	1	-2.67	0	0	40
Max Z	0	0	5.33	0	0	280

The optimal solution :

x_1=80   , $x_2 = 40$, $S_1 = 0$ & $S_2 = 0$ // Z=280

Example 3:

$\qquad$ Use the simplex method to solve the (LP) model:

$$min \ Z = -6x_1 - 10x_2 - 4x_3$$

$\qquad$ Subject to

$$x_1 + x_2 + x_3 \le 1000$$

$$x_1 + x_2 \qquad \le 500$$

$$x_1 + 2x_2 \qquad \le 700$$

$$x_1, x_2, x_3 \qquad \ge 0$$

12.1 **Duality in LPP**

Every LPP called the **primal** is associated with another LPP called **dual**. Either of the problems is primal with the other one as dual. The optimal solution of either problem reveals the information about the optimal solution of the other.

Let the primal problem be

Max $Z_x = c_1x_1 + c_2x_2 + \ldots + c_nx_n$
Subject to restrictions
$\qquad a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$
$\qquad a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$
$\qquad .$
$\qquad .$
$\qquad .$
$\qquad a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_n$
and
$\qquad x_1 \geq 0,\ x_2 \geq 0,\ldots,\ x_n \geq 0$

The corresponding dual is defined as

Min $Z_w = b_1w_1 + b_2w_2 + \ldots + b_mw_m$
Subject to restrictions
$\qquad a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m \geq c_1$
$\qquad a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m \geq c_2$
$\qquad .$
$\qquad .$
$\qquad .$
$\qquad a_{1n}w_1 + a_{2n}w_2 + \ldots\ldots + a_{mn}w_m \geq c_n$
and
$\qquad w_1,\ w_2,\ \ldots,\ w_m \geq 0$

Matrix Notation
Primal
$\qquad$ Max $Z_x = CX$
$\qquad$ Subject to
$\qquad\qquad AX \leq b$ and $X \geq 0$

Dual
$\qquad$ Min $Z_w = b^T W$
$\qquad$ Subject to

$$A^T W \geq C^T \text{ and } W \geq 0$$

12.2 <u>Important characteristics of Duality</u>

1. Dual of dual is primal
2. If either the primal or dual problem has a solution then the other also has a solution and their optimum values are equal.
3. If any of the two problems has an infeasible solution, then the value of the objective function of the other is unbounded.
4. The value of the objective function for any feasible solution of the primal is less than the value of the objective function for any feasible solution of the dual.
5. If either the primal or dual has an unbounded solution, then the solution to the other problem is infeasible.
6. If the primal has a feasible solution, but the dual does not have then the primal will not have a finite optimum solution and vice versa.

12.3 <u>Advantages and Applications of Duality</u>

1. Sometimes dual problem solution may be easier than primal solution, particularly when the number of decision variables is considerably less than slack / surplus variables.
2. In the areas like economics, it is highly helpful in obtaining future decision in the activities being programmed.
3. In physics, it is used in parallel circuit and series circuit theory.
4. In game theory, dual is employed by column player who wishes to minimize his maximum loss while his opponent i.e. Row player applies primal to maximize his minimum gains. However, if one problem is solved, the solution for other also can be obtained from the simplex tableau.
5. When a problem does not yield any solution in primal, it can be verified with dual.
6. Economic interpretations can be made and shadow prices can be determined enabling the managers to take further decisions.

12.4 <u>Steps for a Standard Primal Form</u>

Step 1 – Change the objective function to Maximization form

Step 2 – If the constraints have an inequality sign '$\geq$' then multiply both sides by -1 and convert the inequality sign to '$\leq$'.

Step 3 – If the constraint has an '=' sign then replace it by two constraints involving the inequalities going in opposite directions. For example $x_1 + 2x_2 = 4$ is written as
$x_1 + 2x_2 \leq 4$
$x_1 + 2x_2 \geq 4$ (using step2) $\rightarrow$ $-x_1 - 2x_2 \leq -4$

Step 4 – Every unrestricted variable is replaced by the difference of two non-negative variables.

Step5 – We get the standard primal form of the given LPP in which.

- o All constraints have '≤' sign, where the objective function is of maximization form.
- o All constraints have '≥' sign, where the objective function is of minimization from.

12.5 <u>Rules for Converting any Primal into its Dual</u>

1. Transpose the rows and columns of the constraint co-efficient.
2. Transpose the co-efficient $(c_1, c_2, \ldots c_n)$ of the objective function and the right side constants $(b_1, b_2, \ldots b_n)$
3. Change the inequalities from '≤' to '≥' sign.
4. Minimize the objective function instead of maximizing it.

12.6 <u>Example Problems</u>

<u>Write the dual of the given problems</u>

Example 1

Min $Z_x = 2x_2 + 5x_3$
Subject to
$\qquad x_1 + x_2 \geq 2$
$\qquad 2x_1 + x_2 + 6x_3 \leq 6$
$\qquad x_1 - x_2 + 3x_3 = 4$
$\qquad x_1, x_2, x_3 \geq 0$

Solution

Primal
$\qquad$ Max $Z_x' = -2x_2 - 5x_3$
$\qquad$ Subject to
$\qquad\qquad -x_1 - x_2 \leq -2$
$\qquad\qquad 2x_1 + x_2 + 6x_3 \leq 6$
$\qquad\qquad x_1 - x_2 + 3x_3 \leq 4$
$\qquad\qquad -x_1 + x_2 - 3x_3 \leq -4$
$\qquad\qquad x_1, x_2, x_3 \geq 0$

Dual
$\qquad$ Min $Z_w = -2w_1 + 6w_2 + 4w_3 - 4w_4$
$\qquad$ Subject to
$\qquad\qquad -w_1 + 2w_2 + w_3 - w_4 \geq 0$
$\qquad\qquad -w_1 + w_2 - w_3 + w_4 \geq -2$
$\qquad\qquad 6w_2 + 3w_3 - 3w_4 \geq -5$
$\qquad\qquad w_1, w_2, w_3, w_4 \geq 0$

Example 2

Min $Z_x = 3x_1 - 2x_2 + 4x_3$
Subject to

$3x_1 + 5x_2 + 4x_3 \geq 7$
$6x_1 + x_2 + 3x_3 \geq 4$
$7x_1 - 2x_2 - x_3 \geq 10$
$x_1 - 2x_2 + 5x_3 \geq 3$
$4x_1 + 7x_2 - 2x_3 \geq 2$
$x_1, x_2, x_3 \geq 0$

Solution
Primal

Max $Z_x' = -3x_1 + 2x_2 - 4x_3$
Subject to

$-3x_1 - 5x_2 - 4x_3 \leq -7$
$-6x_1 - x_2 - 3x_3 \leq -4$
$-7x_1 + 2x_2 + x_3 \leq -10$
$-x_1 + 2x_2 - 5x_3 \leq -3$
$-4x_1 - 7x_2 + 2x_3 \leq -2$
$x_1, x_2, x_3 \geq 0$

Dual

Min $Z_w = -7w_1 - 4w_2 - 10w_3 - 3w_4 - 2w_5$
Subject to

$-3w_1 - 6w_2 - 7w_3 - w_4 - 4w_5 \geq -3$
$-5w_1 - w_2 + 2w_3 + 2w_4 - 7w_5 \geq 2$
$-4w_1 - 3w_2 + w_3 - 5w_4 + 2w_5 \geq -4$
$w_1, w_2, w_3, w_4, w_5 \geq 0$

Example 3
Max $Z = 2x_1 + 3x_2 + x_3$
Subject to

$4x_1 + 3x_2 + x_3 = 6$
$x_1 + 2x_2 + 5x_3 = 4$
$x_1, x_2 \geq 0$

Solution

Primal

Max $Z_x = 2x_1 + 3x_2 + x_3$
Subject to

$4x_1 + 3x_2 + x_3 \leq 6$
$-4x_1 - 3x_2 - x_3 \leq -6$
$x_1 + 2x_2 + 5x_3 \leq 4$
$-x_1 - 2x_2 - 5x_3 \leq -4$

$$x_1, x_2 \geq 0$$

Dual

$$\text{Min } Z_w = 6w_1 - 6w_2 + 4w_3 - 4w_4$$
Subject to
$$4w_1 - 4w_2 + w_3 - w_4 \geq 2$$
$$3w_1 - 3w_2 + 2w_3 - 2w_4 \geq 3$$
$$w_1 - w_2 + 5w_3 - 5w_4 \geq 1$$
$$w_1, w_2, w_3, w_4 \geq 0$$

Example 4

$\text{Min } Z_x = x_1 + x_2 + x_3$
Subject to
$$x_1 - 3x_2 + 4x_3 = 5$$
$$x_1 - 2x_2 \leq 3$$
$$2x_2 - x_3 \geq 4$$
$$x_1, x_2 \geq 0, x_3 \text{ is unrestricted in sign}$$

Solution

Primal

$$\text{Max } Z' = -x_1 - x_2 - x_3' + x_3''$$
Subject to
$$x_1 - 3x_2 + 4(x_3' - x_3'') \leq 5$$
$$-x_1 + 3x_2 - 4(x_3' - x_3'') \leq -5$$
$$x_1 - 2x_2 \leq 3$$
$$-2x_2 + x_3' - x_3'' \leq -4$$
$$x_1, x_2, x_3', x_3'' \geq 0$$

Dual

$$\text{Min } Z_w = 5w_1 - 5w_2 + 3w_3 - 4w_4$$
Subject to
$$w_1 - w_2 + w_3 \geq -1$$
$$-3w_1 + 3w_2 - 2w_3 - 2w_4 \geq -1$$
$$4w_1 - 4w_2 + w_4 \geq -1$$
$$-4w_1 + 4w_2 - w_4 \geq 1$$
$$w_1, w_2, w_3, w_4, \geq 0$$

6.5 Finding the Dual of an LP

Associated with any LP is another LP, called the **dual.** Knowing the relation between an LP and its dual is vital to understanding advanced topics in linear and nonlinear programming. This relation is important because it gives us interesting economic insights. Knowledge of duality will also provide additional insights into sensitivity analysis.

In this section, we explain how to find the dual of any LP; in Section 6.6, we discuss the economic interpretation of the dual; and in Sections 6.7–6.10, we discuss the relation that exists between an LP and its dual.

When taking the dual of a given LP, we refer to the given LP as the **primal.** If the primal is a max problem, then the dual will be a min problem, and vice versa. For convenience, we define the variables for the max problem to be $z, x_1, x_2, \ldots, x_n$ and the variables for the min problem to be $w, y_1, y_2, \ldots, y_m$. We begin by explaining how to find the dual of a max problem in which all variables are required to be nonnegative and all constraints are $\leq$ constraints (called a **normal max problem**). A normal max problem may be written as

$$
\begin{aligned}
\max z = {}& c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\
& \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\
& x_j \geq 0 \quad (j = 1, 2, \ldots, n)
\end{aligned}
\tag{16}
$$

The dual of a normal max problem such as (16) is defined to be

$$
\begin{aligned}
\min w = {}& b_1y_1 + b_2y_2 + \cdots + b_mx_m \\
\text{s.t.} \quad & a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1 \\
& a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2 \\
& \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\
& a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n \\
& y_i \geq 0 \quad (i = 1, 2, \ldots, m)
\end{aligned}
\tag{17}
$$

A min problem such as (17) that has all $\geq$ constraints and all variables nonnegative is called a **normal min problem.** If the primal is a normal min problem such as (17), then we define the dual of (17) to be (16).

Finding the Dual of a Normal Max or Min Problem

A tabular approach makes it easy to find the dual of an LP. If the primal is a normal max problem, then it can be read across (Table 14); the dual is found by reading down. Similarly, if the primal is a normal min problem, we find it by reading down; the dual is found

TABLE **14**
Finding the Dual of a Normal Max or Min Problem

min w		max z				
		$(x_1 \geq 0)$	$(x_2 \geq 0)$	$\cdots$	$(x_n \geq 0)$	
		x_1	x_2		x_n	
$(y_1 \geq 0)$	y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$\leq b_1$
$(y_2 \geq 0)$	y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$\leq b_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$(y_m \geq 0)$	y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$\leq b_m$
		$\geq c_1$	$\geq c_2$		$\geq c_n$	

by reading across in the table. We illustrate the use of the table by finding the dual of the Dakota problem and the dual of the diet problems. The Dakota problem is

$$\max z = 60x_1 + 30x_2 + 20x_3$$
$$\begin{aligned}
\text{s.t.} \quad 8x_1 + 6x_2 + x_3 &\leq 48 \quad \text{(Lumber constraint)} \\
4x_1 + 2x_2 + 1.5x_3 &\leq 20 \quad \text{(Finishing constraint)} \\
2x_1 + 1.5x_2 + 0.5x_3 &\leq 8 \quad \text{(Carpentry constraint)} \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

where

$$x_1 = \text{number of desks manufactured}$$
$$x_2 = \text{number of tables manufactured}$$
$$x_3 = \text{number of chairs manufactured}$$

Using the format of Table 14, we read the Dakota problem across in Table 15. Then, reading down, we find the Dakota dual to be

$$\min w = 48y_1 + 20y_2 + 8y_3$$
$$\begin{aligned}
\text{s.t.} \quad 8y_1 + 4y_2 + 2y_3 &\geq 60 \\
6y_1 + 2y_2 + 1.5y_3 &\geq 30 \\
y_1 + 1.5y_2 + 0.5y_3 &\geq 20 \\
y_1, y_2, y_3 &\geq 0
\end{aligned}$$

The tabular method of finding the dual makes it clear that the ith dual *constraint* corresponds to the ith primal *variable* x_i. For example, the first dual constraint corresponds to x_1 (desks), because each number comes from the x_1 (desk) column of the primal. Simi-

TABLE **15**
Finding the Dual of the Dakota Problem

min w		max z			
		$(x_1 \geq 0)$	$(x_2 \geq 0)$	$(x_3 \geq 0)$	
		x_1	x_2	x_3	
$(y_1 \geq 0)$	y_1	8	6	1	≤ 48
$(y_2 \geq 0)$	y_2	4	2	1.5	≤ 20
$(y_3 \geq 0)$	y_3	2	1.5	0.5	≤ 8
		≥ 60	≥ 30	≥ 20	

larly, the second dual constraint corresponds to x_2 (tables), and the third dual constraint corresponds to x_3 (chairs). In a similar fashion, dual *variable* y_i is associated with the *i*th primal constraint. For example, y_1 is associated with the first primal constraint (lumber constraint), because each coefficient of y_1 in the dual comes from the lumber constraint, or the availability of lumber. The importance of these correspondences between the primal and the dual will become clear in Section 6.6.

We now find the dual of the diet problem. Because the diet problem is a min problem, we follow the convention of using w to denote the objective function and y_1, y_2, y_3, and y_4 for the variables. Then the diet problem may be written as

$$\min w = 50y_1 + 20y_2 + 30y_3 + 80y_4$$

$$\begin{array}{lll}
\text{s.t.} & 400y_1 + 200y_2 + 150y_3 + 500y_4 \geq 500 & \text{(Calorie constraint)} \\
& 3y_1 + 2y_2 \geq 6 & \text{(Chocolate constraint)} \\
& 2y_1 + 2y_2 + 4y_3 + 4y_4 \geq 10 & \text{(Sugar constraint)} \\
& 2y_1 + 4y_2 + y_3 + 5y_4 \geq 8 & \text{(Fat constraint)} \\
& y_1, y_2, y_3, y_4 \geq 0 &
\end{array}$$

where

$$y_1 = \text{number of brownies eaten daily}$$
$$y_2 = \text{number of scoops of chocolate ice cream eaten daily}$$
$$y_3 = \text{bottles of soda drunk daily}$$
$$y_4 = \text{pieces of pineapple cheesecake eaten daily}$$

The primal is a normal min problem, so we can read it down, and read its dual across, in Table 16. We find that the dual of the diet problem is

$$\max z = 500x_1 + 6x_2 + 10x_3 + 8x_4$$

$$\begin{array}{ll}
\text{s.t.} & 400x_1 + 3x_2 + 2x_3 + 2x_4 \leq 50 \\
& 200x_1 + 2x_2 + 2x_3 + 4x_4 \leq 20 \\
& 150x_1 + 4x_3 + x_4 \leq 30 \\
& 500x_1 + 4x_3 + 5x_4 \leq 80 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}$$

As in the Dakota problem, we see that the *i*th dual constraint corresponds to the *i*th primal variable. For example, the third dual constraint may be thought of as the soda constraint. Also, the *i*th dual variable corresponds to the *i*th primal constraint. For example, x_3 (the third dual variable) may be thought of as the dual sugar variable.

TABLE 16
Finding the Dual of the Diet Problem

min w		max z				
		$(x_1 \geq 0)$	$(x_2 \geq 0)$	$(x_3 \geq 0)$	$(x_4 \geq 0)$	
		x_1	x_2	x_3	x_4	
$(y_1 \geq 0)$	y_1	400	3	2	2	≤ 50
$(y_2 \geq 0)$	y_2	200	2	2	4	≤ 20
$(y_3 \geq 0)$	y_3	150	0	4	1	≤ 30
$(y_4 \geq 0)$	y_4	500	0	4	5	≤ 80
		≥ 500	≥ 6	≥ 10	≥ 8	

Finding the Dual of a Nonnormal LP

Unfortunately, many LPs are not normal max or min problems. For example,

$$\max z = 2x_1 + x_2$$
$$\text{s.t.} \quad x_1 + x_2 = 2$$
$$2x_1 - x_2 \geq 3 \qquad (18)$$
$$x_1 - x_2 \leq 1$$
$$x_1 \geq 0, x_2 \text{ urs}$$

is not a normal max problem because it has a $\geq$ constraint, an equality constraint, and an unrestricted-in-sign variable. As another example of a nonnormal LP, consider

$$\min w = 2y_1 + 4y_2 + 6y_3$$
$$\text{s.t.} \quad y_1 + 2y_2 + y_3 \geq 2$$
$$y_1 \quad\quad - y_3 \geq 1$$
$$y_2 + y_3 = 1 \qquad (19)$$
$$2y_1 + y_2 \quad\quad \leq 3$$
$$y_1 \text{ urs}, y_2, y_3 \geq 0$$

This LP is not a normal min problem because it contains an equality constraint, a $\leq$ constraint, and an unrestricted-in-sign variable.

Fortunately, an LP can be transformed into normal form (either (16) or (17)). To place a max problem into normal form, we proceed as follows:

Step 1   Multiply each $\geq$ constraint by -1, converting it into a $\leq$ constraint. For example, in (18), $2x_1 - x_2 \geq 3$ would be transformed into $-2x_1 + x_2 \leq -3$.

Step 2   Replace each equality constraint by two inequality constraints (a $\leq$ constraint and a $\geq$ constraint). Then convert the $\geq$ constraint to a $\leq$ constraint. For example, in (18), we would replace $x_1 + x_2 = 2$ by the two inequalities $x_1 + x_2 \geq 2$ and $x_1 + x_2 \leq 2$. Then we would convert $x_1 + x_2 \geq 2$ to $-x_1 - x_2 \leq -2$. The net result is that $x_1 + x_2 = 2$ is replaced by the two inequalities $x_1 + x_2 \leq 2$ and $-x_1 - x_2 \leq -2$.

Step 3   As in Section 4.14, replace each urs variable x_i by $x_i = x' - x_i''$, where $x_i' \geq 0$ and $x_i'' \geq 0$. In (18), we would replace x_2 by $x_2' - x_2''$.

After these transformations are complete, (18) has been transformed into the following (equivalent) LP:

$$\max z = 2x_1 + x_2' - x_2''$$
$$\text{s.t.} \quad x_1 + x_2' - x_2'' \leq 2$$
$$-x_1 - x_2' + x_2'' \leq -2$$
$$-2x_1 + x_2' - x_2'' \leq -3 \qquad (18')$$
$$x_1 - x_2' + x_2'' \leq 1$$
$$x_1, x_2', x_2'' \geq 0$$

Because (18′) is a normal max problem, we could use (16) and (17) to find the dual of (18′).

If the primal is not a normal min problem, then we can transform it into a normal min problem as follows:

Step 1 Convert each $\leq$ constraint into a $\geq$ constraint by multiplying through by -1. For example, in (19), $2y_1 + y_2 \leq 3$ is transformed into $-2y_1 - y_2 \geq -3$.

Step 2 Replace each equality constraint by a $\leq$ constraint and a $\geq$ constraint. Then transform the $\leq$ constraint into a $\geq$ constraint. For example, in (19), the constraint $y_2 + y_3 = 1$ is equivalent to $y_2 + y_3 \leq 1$ and $y_2 + y_3 \geq 1$. Transforming $y_2 + y_3 \leq 1$ into $-y_2 - y_3 \geq -1$, we see that we can replace the constraint $y_2 + y_3 = 1$ by the two constraints $y_2 + y_3 \geq 1$ and $-y_2 - y_3 \geq -1$.

Step 3 Replace any urs variable y_i by $y_i = y_i' - y_i''$, where $y_i' \geq 0$ and $y_i'' \geq 0$. Applying these steps to (19) yields the following standard min problem:

$$
\begin{aligned}
\min w = 2y_1' &- 2y_1'' + 4y_2 + 6y_3 \\
\text{s.t.} \quad y_1' &- y_1'' + 2y_2 + y_3 \geq 2 \\
y_1' &- y_1'' \qquad\quad - y_3 \geq 1 \\
&\quad\; y_2 + y_3 \geq 1 \\
&\; - y_2 - 6y_3 \geq -1 \\
-2y_1' + 2y_1'' &- y_2 \qquad\quad \geq -3 \\
y_1', y_1'', y_2, y_3 &\geq 0
\end{aligned}
\tag{19'}
$$

Because (19') is a normal min problem in standard form, we may use (16) and (17) to find its dual.

We can find the dual of a nonnormal LP without going through the transformations that we have described by using the following rules.[†]

Finding the Dual of a Nonnormal Max Problem

Step 1 Fill in Table 14 so that the primal can be read across.

Step 2 After making the following changes, the dual can be read down in the usual fashion: (a) If the ith primal constraint is a $\geq$ constraint, then the corresponding dual variable y_i must satisfy $y_i \leq 0$. (b) If the ith primal constraint is an equality constraint, then the dual variable y_i is now unrestricted in sign. (c) If the ith primal variable is urs, then the ith dual constraint will be an equality constraint.

When this method is applied to (18), the Table 14 format yields Table 17. We note with an asterisk (*) the places where the rules must be used to determine part of the dual. For example, x_2 urs causes the second dual constraint to be an equality constraint. Also, the first primal constraint being an equality constraint makes y_1 urs, and the second primal constraint being a $\geq$ constraint makes $y_2 \leq 0$. Filling in the missing information across from the appropriate asterisk yields Table18. Reading the dual down, we obtain

$$
\begin{aligned}
\min w = 2y_1 &+ 3y_2 + y_3 \\
\text{s.t.} \quad y_1 &+ 2y_2 + y_3 \geq 2 \\
y_1 &- y_2 - y_3 = 1 \\
y_1 \text{ urs}, \; &y_2 \leq 0, \; y_3 \geq 0
\end{aligned}
$$

In Section 6.8, we give an intuitive explanation of why an equality constraint yields an unrestricted-in-sign dual variable and why a $\geq$ constraint yields a negative dual variable.

We can use the following rules to take the dual of a nonnormal min problem.

[†]In Problems 5 and 6 at the end of this section, we show that these rules are consistent with taking the dual of the transformed LP via (16) and (17).

TABLE **17**

Finding the Dual of LP (18)

		max z		
min w		$(x_1 \geq 0)$	$(x_2$ urs$)^*$	
		x_1	x_2	
	y_1	1	1	$=2^*$
	y_2	2	-1	$\geq 3^*$
$(y_3 \geq 0)$	y_3	1	-1	≤ 1
		≥ 2	$=1$	

TABLE **18**

Finding the Dual of LP (18) (Continued)

		max z		
min w		$(x_1 \geq 0)$	$(x_2$ urs$)$	
		x_1	x_2	
$(y_1$ urs$)$	y_1	1	1	$=2$
$(y_2 \leq 0)$	y_2	2	-1	≥ 3
$(y_3 \geq 0)$	y_3	1	-1	≤ 1
		≥ 2	$=1$	

Finding the Dual of a Nonnormal Min Problem

Step 1   Write out the primal so it can be read down in Table 14.

Step 2   Except for the following changes, the dual can be read across the table: (a) If the ith primal constraint is a $\leq$ constraint, then the corresponding dual variable x_i must satisfy $x_i \leq 0$. (b) If the ith primal constraint is an equality constraint, then the corresponding dual variable x_i will be urs. (c) If the ith primal variable y_i is urs, then the ith dual constraint is an equality constraint.

When this method is applied to (19), we get Table 19. Asterisks (*) show where the new rules must be used to determine parts of the dual. Because y_1 is urs, the first dual constraint is an equality. The third primal constraint is an equality, so dual variable x_3 is urs. Finally, because the fourth primal constraint is a $\leq$ constraint, the fourth dual variable x_4 must satisfy $x_4 \leq 0$. We can now complete the table (see Table 20). Reading the dual across, we obtain

TABLE **19**

Finding the Dual of LP (19)

		max z				
min w		$(x_1 \geq 0)$	$(x_2 \geq 0)$			
		x_1	x_2	x_3	x_4	
$(y_1$ urs$)^*$	y_1	1	1	0	2	2
$(y_2 \geq 0)$	y_2	2	0	1	1	≤ 4
$(y_3 \geq 0)$	y_3	1	-1	1	0	≤ 6
		≥ 2	≥ 1	$=1^*$	$\leq 3^*$	

TABLE **20**
Finding the Dual of LP (19) (Continued)

min w		max z				
		$(x_1 \geq 0)$	$(x_2 \geq 0)$	$(x_3 \geq \text{urs})$	$(x_4 \leq 0)$	
		x_1	x_2	x_3	x_4	
$(y_1 \text{ urs})$	y_1	1	1	0	2	$=2$
$(y_2 \geq 0)$	y_2	2	0	1	1	≤ 4
$(y_3 \geq 0)$	y_3	1	-1	1	0	≤ 6
		≥ 2	≥ 1	$=1$	≤ 3	

$$\max z = 2x_1 + x_2 + x_3 + 3x_4$$
$$\text{s.t.} \quad x_1 + x_2 \qquad + 2x_4 = 2$$
$$2x_1 \qquad + x_3 + x_4 \leq 4$$
$$x_1 - x_2 + x_3 \qquad \leq 6$$
$$x_1, x_2 \geq 0, x_3 \text{ urs}, x_4 \leq 0$$

The reader may verify that with these rules, the dual of the dual is always the primal. This is easily seen from the Table 14 format, because when you take the dual of the dual you are changing the LP back to its original position.

Special Cases of Linear Programming

In this section, we encounter three types of LPs that do not have unique optimal solutions.

1 Some LPs have an infinite number of optimal solutions (*alternative* or *multiple opti-mal solutions*).

2 Some LPs have no feasible solutions (*infeasible* LPs).

3 Some LPs are *unbounded:* There are points in the feasible region with arbitrarily large (in a max problem) z-values.

Alternative or Multiple Optimal Solutions

EXAMPLE 3

An auto company manufactures cars and trucks. Each vehicle must be processed in the paint shop and body assembly shop. If the paint shop were only painting trucks, then 40 per day could be painted. If the paint shop were only painting cars, then 60 per day could be painted. If the body shop were only producing cars, then it could process 50 per day. If the body shop were only producing trucks, then it could process 50 per day. Each truck contributes $300 to profit, and each car contributes $200 to profit. Use linear program-ming to determine a daily production schedule that will maximize the company's profits.

Solution The company must decide how many cars and trucks should be produced daily. This leads us to define the following decision variables:

$$x_1 = \text{number of trucks produced daily}$$
$$x_2 = \text{number of cars produced daily}$$

The company's daily profit (in hundreds of dollars) is $3x_1 + 2x_2$, so the company's objective function may be written as

$$\max z = 3x_1 + 2x_2 \qquad (12)$$

The company's two constraints are the following:

Constraint 1 The fraction of the day during which the paint shop is busy is less than or equal to 1.

Constraint 2 The fraction of the day during which the body shop is busy is less than or equal to 1.

We have

$$\text{Fraction of day paint shop works on trucks} = \left(\frac{\text{fraction of day}}{\text{truck}}\right)\left(\frac{\text{trucks}}{\text{day}}\right)$$

$$= \tfrac{1}{40}\, x_1$$

$$\text{Fraction of day paint shop works on cars} = \tfrac{1}{60}\, x_2$$

$$\text{Fraction of day body shop works on trucks} = \tfrac{1}{50}\, x_1$$

$$\text{Fraction of day body shop works on cars} = \tfrac{1}{50}\, x_2$$

Thus, Constraint 1 may be expressed by

$$\frac{1}{40}\, x_1 + \frac{1}{60}\, x_2 \le 1 \qquad \text{(Paint shop constraint)} \qquad (13)$$

and Constraint 2 may be expressed by

$$\frac{1}{50}\, x_1 + \frac{1}{50}\, x_2 \le 1 \qquad \text{(Body shop constraint)} \qquad (14)$$

Because $x_1 \ge 0$ and $x_2 \ge 0$ must hold, the relevant LP is

$$\max z = 3x_1 + 2x_2 \qquad (12)$$

$$\text{s.t.} \quad \frac{1}{40}\, x_1 + \frac{1}{60}\, x_2 \le 1 \qquad (13)$$

$$\frac{1}{50}\, x_1 + \frac{1}{50}\, x_2 \le 1 \qquad (14)$$

$$x_1, x_2 \ge 0$$

The feasible region for this LP is the shaded region in Figure 5 bounded by *AEDF*.[†]

For our isoprofit line, we choose the line passing through the point (20, 0). Because (20, 0) has a z-value of $3(20) + 2(0) = 60$, this yields the isoprofit line $z = 3x_1 + 2x_2 = 60$. Examining lines parallel to this isoprofit line in the direction of increasing z (northeast), we find that the last "point" in the feasible region to intersect an isoprofit line is the *entire* line segment *AE*. This means that any point on the line segment *AE* is optimal. We can use any point on *AE* to determine the optimal z-value. For example, point *A*, (40, 0), gives $z = 3(40) = 120$.

In summary, the auto company's LP has an infinite number of optimal solutions, or *multiple* or *alternative optimal solutions*. This is indicated by the fact that as an isoprofit

[†]Constraint (13) is satisfied by all points on or below *AB* (*AB* is $\tfrac{1}{40}\, x_1 + \tfrac{1}{60}\, x_2 = 1$), and (14) is satisfied by all points on or below *CD* (*CD* is $\tfrac{1}{50}\, x_1 + \tfrac{1}{50}\, x_2 = 1$).
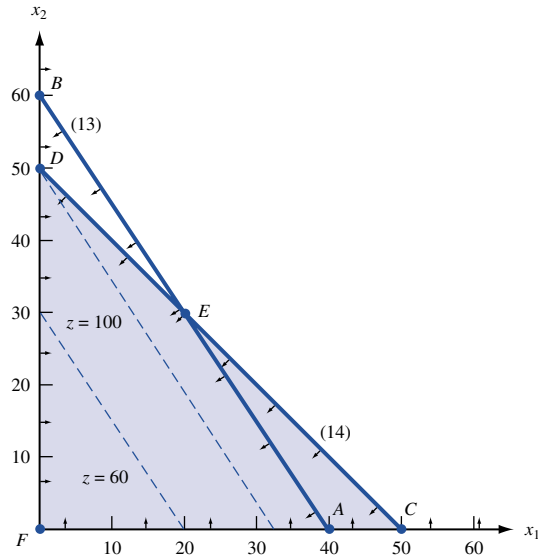
FIGURE 5
Graphical Solution of
Example 3

line leaves the feasible region, it will intersect an entire line segment corresponding to the binding constraint (in this case, AE).

From our current example, it seems reasonable (and can be shown to be true) that if two points (A and E here) are optimal, then *any* point on the line segment joining these two points will also be optimal.

If an alternative optimum occurs, then the decision maker can use a secondary criterion to choose between optimal solutions. The auto company's managers might prefer point A because it would simplify their business (and still allow them to maximize profits) by allowing them to produce only one type of product (trucks).

The technique of **goal programming** (see Section 4.14) is often used to choose among alternative optimal solutions.

Infeasible LP

It is possible for an LP's feasible region to be empty (contain no points), resulting in an *infeasible* LP. Because the optimal solution to an LP is the best point in the feasible region, an infeasible LP has no optimal solution.
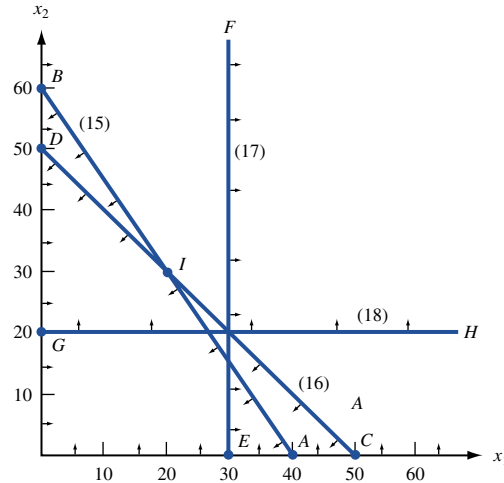
EXAMPLE 4

Suppose that auto dealers require that the auto company in Example 3 produce at least 30 trucks and 20 cars. Find the optimal solution to the new LP.

Solution   After adding the constraints $x_1 \geq 30$ and $x_2 \geq 20$ to the LP of Example 3, we obtain the following LP:

$$\max z = 3x_1 + 2x_2$$

$$\text{s.t.} \quad \frac{1}{40} x_1 + \frac{1}{60} x_2 \leq 1 \tag{15}$$

$$\frac{1}{50} x_1 + \frac{1}{50} x_2 \leq 1 \tag{16}$$

FIGURE 6
An Empty Feasible Region (Infeasible LP)

$$x_1 \qquad \geq 30 \qquad\qquad \text{(17)}$$
$$x_2 \geq 20 \qquad\qquad \text{(18)}$$
$$x_1, x_2 \geq 0$$

The graph of the feasible region for this LP is Figure 6.

Constraint (15) is satisfied by all points on or below *AB* (*AB* is $\frac{1}{40}x_1 + \frac{1}{60}x_2 = 1$).

Constraint (16) is satisfied by all points on or below *CD* (*CD* is $\frac{1}{50}x_1 + \frac{1}{50}x_2 = 1$).

Constraint (17) is satisfied by all points on or to the right of *EF* (*EF* is $x_1 = 30$).

Constraint (18) is satisfied by all points on or above *GH* (*GH* is $x_2 = 20$).

From Figure 6 it is clear that no point satisfies all of (15)–(18). This means that Example 4 has an empty feasible region and is an infeasible LP.

In Example 4, the LP is infeasible because producing 30 trucks and 20 cars requires more paint shop time than is available.

Unbounded LP

Our next special LP is an *unbounded* LP. For a max problem, an unbounded LP occurs if it is possible to find points in the feasible region with arbitrarily large z-values, which corresponds to a decision maker earning arbitrarily large revenues or profits. This would indicate that an unbounded optimal solution should not occur in a correctly formulated LP. Thus, if the reader ever solves an LP on the computer and finds that the LP is unbounded, then an error has probably been made in formulating the LP or in inputting the LP into the computer.

For a minimization problem, an LP is unbounded if there are points in the feasible region with arbitrarily small z-values. When graphically solving an LP, we can spot an unbounded LP as follows: A max problem is unbounded if, when we move parallel to our original isoprofit line in the direction of increasing z, we never entirely leave the feasible region. A minimization problem is unbounded if we never leave the feasible region when moving in the direction of decreasing z.

Graphically solve the following LP:

$$\max z = 2x_1 - x_2$$
$$\text{s.t.} \qquad x_1 - x_2 \le 1 \qquad\qquad (19)$$
$$2x_1 + x_2 \ge 6 \qquad\qquad (20)$$
$$x_1, x_2 \ge 0$$

Solution From Figure 7, we see that (19) is satisfied by all points on or above AB (AB is the line $x_1 - x_2 = 1$). Also, (20) is satisfied by all points on or above CD (CD is $2x_1 + x_2 = 6$). Thus, the feasible region for Example 5 is the (shaded) unbounded region in Figure 7, which is bounded only by the x_2 axis, line segment DE, and the part of line AB beginning at E. To find the optimal solution, we draw the isoprofit line passing through $(2, 0)$. This isoprofit line has $z = 2x_1 - x_2 = 2(2) - 0 = 4$. The direction of increasing z is to the southeast (this makes x_1 larger and x_2 smaller). Moving parallel to $z = 2x_1 - x_2$ in a southeast direction, we see that any isoprofit line we draw will intersect the feasible region. (This is because any isoprofit line is steeper than the line $x_1 - x_2 = 1$.)

    Thus, there are points in the feasible region that have arbitrarily large z-values. For example, if we wanted to find a point in the feasible region that had $z \ge 1{,}000{,}000$, we could choose any point in the feasible region that is southeast of the isoprofit line $z = 1{,}000{,}000$.

    From the discussion in the last two sections, we see that every LP with two variables must fall into one of the following four cases:

Case 1 The LP has a unique optimal solution.

Case 2 The LP has alternative or multiple optimal solutions: Two or more extreme points are optimal, and the LP will have an infinite number of optimal solutions.

Case 3 The LP is infeasible: The feasible region contains no points.

Case 4 The LP is unbounded: There are points in the feasible region with arbitrarily large z-values (max problem) or arbitrarily small z-values (min problem).

In Chapter 4, we show that every LP (not just LPs with two variables) must fall into one of Cases 1–4.
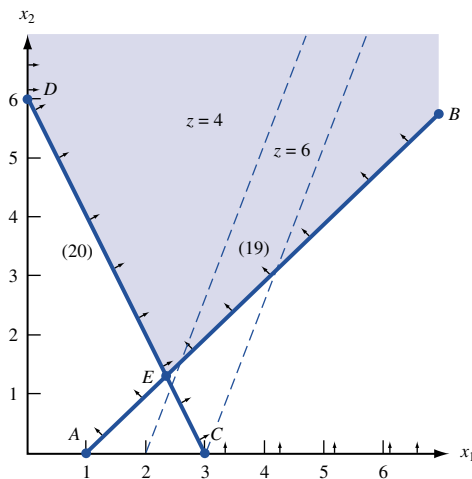

FIGURE 7
An Unbounded LP

The Transportation Problem

[ ]

Formulating Transportation Problems

Example

PowerCo has three electricity generating plants to supply the electrical demands of four cities. Each plant can supply the following maximum output (in kilowatt-hours [kwh]) of electricity:

- **Plant 1**: 35 million kwh.

- **Plant 2**: 50 million kwh.

- **Plant 3**: 40 million kwh.

The peak power demand for each city is as follows:

- **City 1**: 45 million kwh.

- **City 2**: 20 million kwh.

- **City 3**: 30 million kwh.

- **City 4**: 30 million kwh.

The cost of transmitting one million kwh from a given plant to a given city depends on the distance between the two, and is defined by the following table:

From	To				
	City 1	**City 2**	**City 3**	**City 4**	Supply
Plant 1	$8	$6	$10	$9	35
Plant 2	$9	$12	$13	$7	50
Plant 3	$14	$9	$16	$5	40
Demand	45	20	30	30	

PowerCo's problem is to work out how to meet each city's peak demand <u>at minimum total cost</u>.

Formulation:

Let x_{ij} be the number of million kwh sent from Plant i to City j, for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$. (Just note, in passing, that the variables in a problem like this naturally have **two** subscripts, instead of **one**.)

Supply constraints:

The total power supplied by each plant cannot exceed its capacity. Therefore:

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 35 \qquad \textbf{(Plant 1)}$$

$$x_{21} + x_{22} + x_{23} + x_{24} \leq 50 \qquad \textbf{(Plant 2)}$$

$$x_{31} + x_{32} + x_{33} + x_{34} \leq 40 \qquad \textbf{(Plant 3)}$$

Demand constraints:

The total power received by each city must meet or exceed its peak demand. Therefore:

$$x_{11} + x_{21} + x_{31} \geq 45 \qquad \textbf{(City 1)}$$

$$x_{12} + x_{22} + x_{32} \geq 20 \qquad \textbf{(City 2)}$$

$$x_{13} + x_{23} + x_{33} \geq 30 \qquad \textbf{(City 3)}$$

$$x_{14} + x_{24} + x_{34} \geq 30 \qquad \textbf{(City 4)}$$

Also, of course, $x_{ij} \geq 0$, for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$. (Note that each variable x_{ij} appears in <u>exactly one</u> supply constraint and <u>exactly one</u> demand constraint.)

Finally, the cost of transmitting all this power is given by

$$\begin{aligned} z \quad = \quad & 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} \\ & +9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} \\ & +13x_{31} + 9x_{32} + 16x_{33} + 5x_{34} \end{aligned}$$

and this quantity is to be <u>minimised</u>.

(For future reference, note that the optimal solution of this problem is given by $x_{12} = 10$, $x_{13} = 25$, $x_{21} = 45$, $x_{23} = 5$, $x_{32} = 10$, $x_{34} = 30$ and $z = 1020$. [Check that it is feasible!])