Written out in full, the example problem above looks like this:

$$\min z = 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} + 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34}$$

$$
\begin{array}{lcccccccccccccr}
\text{s.t.} & x_{11} & +x_{12} & +x_{13} & +x_{14} & & & & & & & & & \leq 35 \\
& & & & & x_{21} & +x_{22} & +x_{23} & +x_{24} & & & & & \leq 50 \\
& & & & & & & & & x_{31} & +x_{32} & +x_{33} & +x_{34} & \leq 40 \\[6pt]
& x_{11} & & & & +x_{21} & & & & +x_{31} & & & & \geq 45 \\
& & x_{12} & & & & +x_{22} & & & & +x_{32} & & & \geq 20 \\
& & & x_{13} & & & & +x_{23} & & & & +x_{33} & & \geq 30 \\
& & & & x_{14} & & & & +x_{24} & & & & +x_{34} & \geq 30 \\[6pt]
& x_{11}, & x_{12}, & x_{13}, & x_{14}, & x_{21}, & x_{22}, & x_{23}, & x_{24}, & x_{31}, & x_{32}, & x_{33}, & x_{34} & \geq 0
\end{array}
$$

It should be clear that this is an LP problem. However, it has some special features that make it worthwhile developing **specialised methods** for its solution (instead of depending on the various kinds of Simplex Method). For example:

- As noted before, each variable $x_{ij}$ appears in exactly one supply constraint and exactly one demand constraint.

- Note the "patterns" exhibited by the set of constraints.

- Within each constraint, every variable either has the coefficient $0$ or $+1$.

- The matrix representing the constraints therefore consists entirely of $0$s and $+1$s, all arranged in a **specific pattern**.

**Fig. 6** shows a graphical representation of the example problem, with the variables $x_{ij}$ represented by lines connecting plants to cities. **Fig. 7** shows the stated optimal solution for the problem (with dashed lines indicating those routes which are not being used to transmit any power).
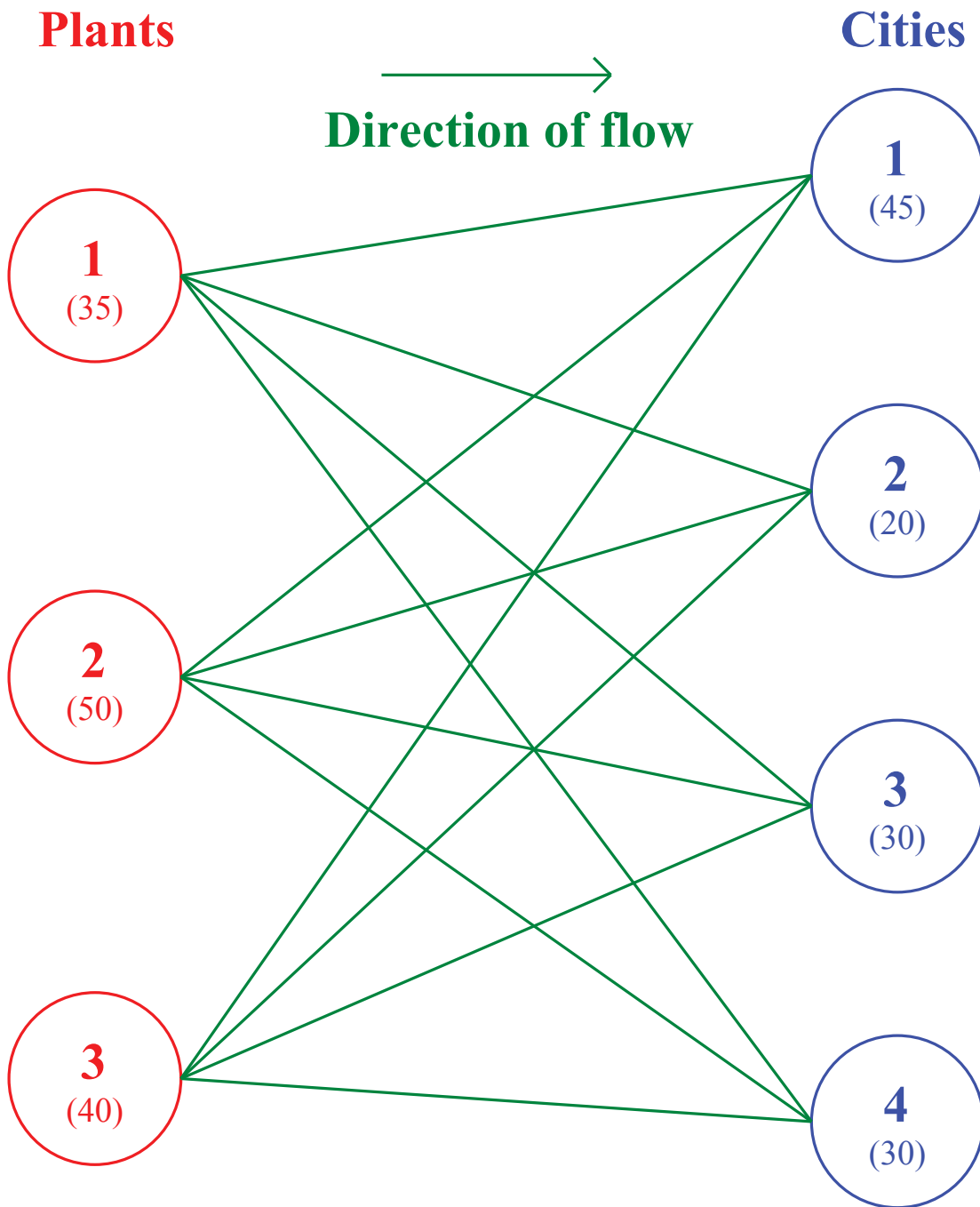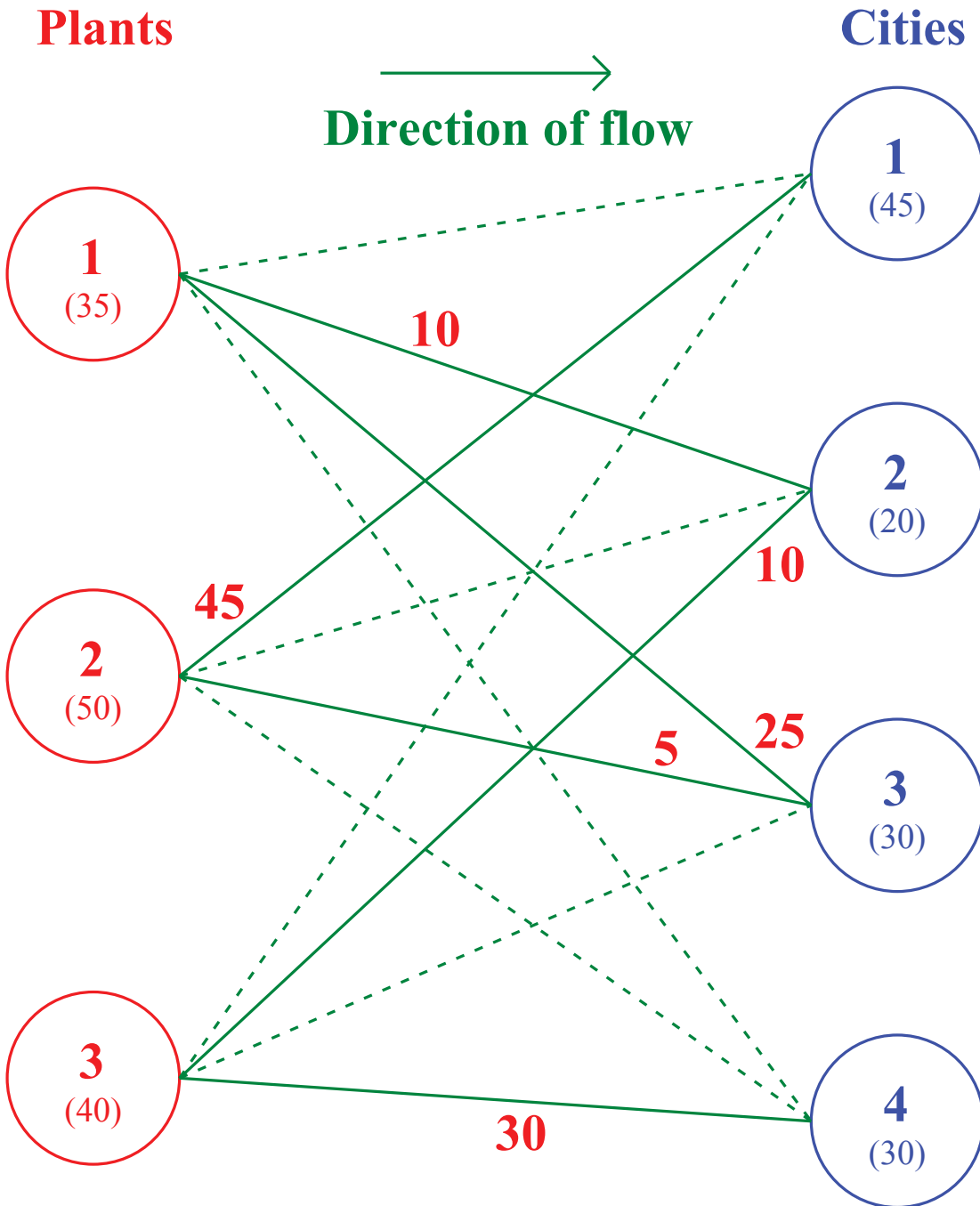
**Fig. 6**

**Fig. 7**

## General Description of a Transportation Problem

A transportation problem (**TP**) is specified by:

- A set of $m$ **supply points** from which some commodity is shipped. Supply point $i$ can provide at most $s_i$ units of the commodity.

- A set of $n$ **demand points** to which the commodity is shipped. Demand point $j$ requires at least $d_j$ units of the commodity.

- Each unit of the commodity shipped from supply point $i$ to demand point $j$ incurs a cost of $c_{ij}$.

- The problem requires the shipping of the commodity from supply points to demand points in such a way as to minimise the total cost of shipping, while meeting all supply and demand restrictions.

Let $x_{ij}$ be the number of units of the commodity shipped from supply point $i$ to demand point $j$. Then the mathematical formulation of the Transportation Problem is:

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} \leq s_i, \quad \text{for } i = 1, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \geq d_j, \quad \text{for } j = 1, \ldots, n$$

$$x_{ij} \geq 0 \ \text{ for } i = 1, \ldots, m \text{ and } j = 1, \ldots, n \ .$$

**Balanced Transportation Problems**

You may have noticed, in the example problem, that the total supply of electricity available exactly equalled the total amount demanded. A general **TP** which satisfies this requirement is called a **Balanced Problem**:

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j \ .$$

In such a case, it immediately follows that all supply constraints and all demand constraints <u>must be satisfied</u> **as equalities**.

(If you are not sure that this is true, think about what would happen if, for example, Plant 1 only supplied 30 million kwh out of its maximum supply capability of 35. Then the other two plants would not have enough total capacity to meet all of the remaining demand, meaning that at least one city's demands would be partially or totally unmet.) Therefore, the mathematical formulation of a **Balanced TP** takes this form:

$$\min \ \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} \ = \ s_i, \ \text{ for } \ i = 1,\ldots,m$$

$$\sum_{i=1}^{m} x_{ij} \ = \ d_j, \ \text{ for } \ j = 1,\ldots,n$$

$$x_{ij} \ \geq \ 0 \ \text{ for } \ i = 1,\ldots,m \text{ and } j = 1,\ldots,n \ .$$

(Later, we will see that it is very easy to find an **initial bfs** for a <u>balanced</u> TP.)

From now on, we will assume that (in any balanced TP)

$$s_i \geq 0 \ \text{ for } \ i = 1, \ldots, m \ , \quad \text{and}$$

$$d_j \geq 0 \ \text{ for } \ j = 1, \ldots, n \ .$$

Clearly, no feasible solutions can exist for a balanced TP if any $s_i < 0$ or any $d_j < 0$ (check the supply and demand constraints).

A TP is completely specified by the supply, the demand and the costs, so this data and any **BFS** may conveniently be represented in a **Transportation Tableau**. The **cell** in row $i$ and column $j$ corresponds to the variable $x_{ij}$. The cost $c_{ij}$ is entered in the sub-cell in the upper-left-hand-corner. If $x_{ij}$ is a basic variable, then its value is entered into the $ij^{th}$ cell (otherwise, the value of $x_{ij}$ is zero).

| | 1 | 2 | $\ldots$ | $n$ | **supply** |
|---|---|---|---|---|---|
| **1** | $c_{11}$ | $c_{12}$ | $\ldots$ | $c_{1n}$ | $s_1$ |
| **2** | $c_{21}$ | $c_{22}$ | $\ldots$ | $c_{2n}$ | $s_2$ |
| $\vdots$ | | | | | $\vdots$ |
| **m** | $c_{m1}$ | $c_{m2}$ | $\ldots$ | $c_{mn}$ | $s_m$ |
| **demand** | $d_1$ | $d_2$ | $\ldots$ | $d_n$ | |

As an example of the Transportation Tableau, here is the optimal solution for the PowerCo problem:

| | 1 | 2 | 3 | 4 | supply |
|---|---|---|---|---|---|
| 1 | 8 | 6 **10** | 10 **25** | 9 | 35 |
| 2 | 9 **45** | 12 | 13 **5** | 7 | 50 |
| 3 | 14 | 9 **10** | 16 | 5 **30** | 40 |
| demand | 45 | 20 | 30 | 30 | |

(Note that it is very easy to check that all the supply and demand constraints are satisfied, just by summing along the rows and down the columns.)

$$
\begin{aligned}
\text{Total cost} &= 6 \times 10 + 10 \times 25 + 9 \times 45 \\
&\quad + 13 \times 5 + 9 \times 10 + 5 \times 30 \\
&= 1020
\end{aligned}
$$

**Balancing an Unbalanced Transportation Problem**

- If

$$\text{total supply} > \text{total demand}$$

  then we just create a **dummy demand point** that has a demand equal to the excess of the supply over the demand. Because shipments to the dummy demand point are not real shipments, they are assigned costs of **zero**. (When the problem is actually solved, shipments to the dummy demand point indicate <u>unused</u> supply capacity.)

- If

$$\text{total supply} < \text{total demand}$$

  then the TP has **no feasible solution**, We can either leave the problem at this point or it may be desirable to explore what happens when we leave some demand unsatisfied by creating a **dummy supply point** with capacity equal to the excess of the demand over the supply. We can assign differing **penalty costs** to shipments from the dummy supply point if we wish to exercise a measure of control over which demand points may have their demands unmet (which will be indicated in the solution by shipments from the dummy supply point).

# Examples

**(1) Excess Supply**: consider the PowerCo problem with the demand for City 1 reduced to 40 million kwh. Then we create a dummy demand point ("**City 5**") with a demand of

(35 + 50 + 40) - (40 + 20 + 30 + 30) = 5 million kwh. From each plant, the cost of shipping to City 5 is set to be zero.

Below is a Transportation Tableau containing the optimal solution for the revised problem:

| | 1 | 2 | 3 | 4 | 5 | supply |
|---|---|---|---|---|---|---|
| **1** | 8 | 6 ⟶ 15 | 10 ⟶ 20 | 9 | 0 | 35 |
| **2** | 9 ⟶ 40 | 12 | 13 ⟶ 10 | 7 | 0 | 50 |
| **3** | 14 | 9 ⟶ 5 | 16 | 5 ⟶ 30 | 0 ⟶ 5 | 40 |
| **demand** | 40 | 20 | 30 | 30 | 5 | |

Because $x_{35} = 5$, we conclude that 5 million kwh of the maximum output of Plant 3 will be unused.

**(2) Excess Demand**: two reservoirs are available to supply the water needs of three cities. Each reservoir can supply up to 50 million gallons of water per day. Each city wishes to receive 40 million gallons per day. For each million gallons of water that is **not** supplied per day, there is a **penalty** (specified by contract):

- For City 1, the penalty is $20.

- For City 2, the penalty is $22.

- For City 3, the penalty is $23.

The cost of transporting a million gallons from each of the reservoirs to each of the cities is given by:

| | To | | |
|---|---|---|---|
| From | City 1 | City 2 | City 3 |
| **Reservoir 1** | $7 | $8 | $10 |
| **Reservoir 2** | $9 | $7 | $8 |

We add a dummy reservoir (**Reservoir 3**) which "supplies" the excess demand of (40 + 40 + 40) - (50 + 50) = 20 million gallons per day. The solution turns out to be:

|  | 1 | 2 | 3 | supply |
|---|---|---|---|---|
| 1 | 7   **20** | 8   **30** | 10 | 50 |
| 2 | 9 | 7   **10** | 8   **40** | 50 |
| 3 | 20   **20** | 22 | 23 | 20 |
| demand | 40 | 40 | 40 | |

City 1 has 20 million gallons of its demand unmet.

## Basic Feasible Solution:

Transportation problem containing **(n + m)** of **structured constraints** and **(n × m)** of **variables**. A feasible solution to a transportation problem is said to be a **basic feasible solution** if the number of basic variables = **(m + n − 1)**

We can find basic feasible solutions using one of the following ways:
**1- North–west corner**

**2- Least cost**

**3- Vogel approximation method (penalty method)**

The three methods differ in the "quality" of the starting basic solution they produce, in the sense that a better starting solution yields a smaller objective value. In general, though not always, the Vogel method yields the best starting basic solution, and the northwest-corner method yields the worst. The tradeoff is that the northwest-corner method involves the least amount of computations.

**1- North-west corner**

The method starts at the northwest-corner cell (route) of the table (variable $x_{11}$).
**Step 1.** Start with cell (1, 1) from the left hand (topmost left corner) and assign it the maximum possible amount.
**Step 2.** Then move to the right-hand cell (1, 2) if there is still any available quantity left, otherwise move to the down cell (2, 1) and assign it maximum possible amount.
**Step 3.** Repeat step 2 and continue until all the available quantity is finished.
**===================================================**

**Example:** Use the north-west corner to find the basic feasible solution for the following transportation problem:
**Solution**:

|  | $D_1$ | $D_2$ | $D_3$ | $D_4$ |  |
|---|---|---|---|---|---|
| $S_1$ | $X_{11}$   10 | $X_{12}$   0 | $X_{13}$   20 | $X_{14}$   11 | 15 |
| $S_2$ | $X_{21}$   12 | $X_{22}$   7 | $X_{23}$   9 | $X_{24}$   20 | 25 |
| $S_3$ | $X_{31}$   0 | $X_{32}$   14 | $X_{33}$   16 | $X_{34}$   18 | 5 |
|  | 5 | 15 | 15 | 10 | 45 / 45 |

$$\sum_{i=1}^{i=m} s_i = \sum_{j=1}^{j=n} d_j = 45$$

|  | 10 | 0 | 20 | 11 |  |
|---|---|---|---|---|---|
| $X_{11}=5$ | | $X_{12}=10$ | $X_{13}=0$ | $X_{14}=0$ | 15 |
| $X_{21}=0$   12 | $X_{22}=5$   7 | $X_{23}=15$   9 | $X_{24}=5$   20 |  | 25 |
| $X_{31}=0$   0 | $X_{32}=0$   14 | $X_{33}=0$   16 | $X_{34}=5$   18 |  | 5 |
|  | 5 | 15 | 15 | 10 | 45 / 45 |

basic variables are:

$X_{11}=5$ , $X_{12}=10$ , $X_{22}=5$ , $X_{23}=15$ , $X_{24}=5$ , $X_{34}=5$

non basic variables is :

$X_{13}=X_{14}=X_{21}=X_{31}=X_{32}=X_{33}=0$

Min.     $Z = \sum_{j=1}^{n}\sum_{i=1}^{m} C_{ij} X_{ij}$

$Z = 5*10+10*0+5*7+15*9+5*20+5*18$
   $= 410$

the number basic variables = m + n − 1 =3+4-1=6
So this solution is basic feasible solution

===========================================

**Example:** Use the north west-corner to find the basic feasible solution for the following transportation problem :

| plants | warehouses | | | | supply |
|---|---|---|---|---|---|
| | $W_1$ | $W_2$ | $W_3$ | $W_4$ | |
| **P1** | 21 | 32 | 52 | 12 | 7000 |
| **P2** | 72 | 32 | 42 | 62 | 9000 |
| **P3** | 42 | 10 | 72 | 22 | 18000 |
| **Demand** | 5000 | 8000 | 7000 | 14000 | |

**Solution**:

$$\sum_{i=1}^{i=m} s_i = \sum_{j=1}^{j=n} d_j = 34000$$

| plants | warehouses | | | | supply |
|---|---|---|---|---|---|
| | $W_1$ | $W_2$ | $W_3$ | $W_4$ | |
| **P1** | 21<br>**5000** | 32<br>**2000** | 52<br>**0** | 12<br>**0** | 7000 |
| **P2** | 72<br>**0** | 32<br>**6000** | 42<br>**3000** | 62<br>**0** | 9000 |
| **P3** | 42<br>**0** | 10<br>**0** | 72<br>**4000** | 22<br>**14000** | 18000 |
| **Demand** | 5000 | 8000 | 7000 | 14000 | 34000 |

Basic variables are: P1W1=5000, P1W2=2000, P2W2= 6000, P2W3=3000, P3W3= 4000, P3W4= 14000

Non basic variables are: P1W3, P1W4, P2W1, P2W4, P3W1, P3W2=0
total cost of the transportation problem is:

$$Min. \quad Z = \sum_{j=1}^{n}\sum_{i=1}^{m} C_{ij} X_{ij}$$

Z= 1083000

The number basic variables = m + n – 1 =3+4-1=6. So this solution is a basic feasible solution.

## 2- Least-Cost Method:

The least-cost method finds a better starting solution by concentrating on the cheapest routes. The method assigns as much as possible to the cell with the smallest unit cost. Next, the satisfied row or column is crossed out and the amounts of supply and demand are adjusted accordingly.

**Step 1:** Choose the cell with the lowest cost and assign it as much as possible. If such a cell of the lowest cost is not unique, select any one of these cells.
**Step 2:** Again, examine the cost matrix and select the cell with the lowest cost (the cell in which the allocation has been made is not considered). Allocate there as much as possible.
**Step 3:** Continue the procedure until all the available quantity is finished.

**Example:** Use the least-cost method to find the solution to the following transportation problem:

| Destinations Source | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $S_1$ | 10 | 2 | 20 | 11 | 15 |
| $S_2$ | 12 | 7 | 9 | 20 | 25 |
| $S_3$ | 4 | 14 | 16 | 18 | 10 |
| Demand | 5 | 15 | 15 | 15 | 50     50 |

**Solution**:

$$\sum_{i=1}^{i=m} s_i = \sum_{j=1}^{j=n} d_j = 50$$

| Destinations Source | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $S_1$ | **0**    10 | **15**    2 | 20    . | 11    . | 15 |

| | | | | | |
|---|---|---|---|---|---|
| S$_2$ | 0 $^{12}$ | . $^{7}$ | 15 $^{9}$ | 10 $^{20}$ | 25 |
| S$_3$ | 5 $^{4}$ | 0 $^{14}$ | 0 $^{16}$ | 5 $^{18}$ | 10 |
| **Demand** | 5 | 15 | 15 | 15 | |

Basic variables are: ………………

Non basic variables are: ………………..

The total cost of the transportation problem is:

$$Min. \quad Z = \sum_{j=1}^{n}\sum_{i=1}^{m} C_{ij} X_{ij}$$

Z= 475

=========================================================

**3- Vogel Approximation Method (VAM)**

**VAM** is an improved version of the least-cost method that generally, but not always, produces better starting solutions.

**Step 1:** Write the differences between the smallest and the second smallest costs in each column below the corresponding column.

**Step 2:** Write the differences between the smallest and the second smallest costs in each row to the right of the corresponding row.
These individual differences are regarded as a **penalty** for making assignments in the second-lowest cost cell in place of the lowest cost cell in each row (or column).

**Step 3:** Select the row or column for which the penalty is the largest and put the maximum possible amount to the cell with the lowest cost in that particular row or column. If there are more than one largest penalty row or column, then select any one of them.

**Step 4:** Cross (or leave) row or column in which the requirement (or demand) has been satisfied and then construct the reduced matrix. If a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).

**Step 5:** Continue this process on the reduced till all the allocations for this method are finished.

**Example1:** Use the **(VAM)** to find the solution to the following problem:

| Destinations<br>Source | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $S_1$ | 10 | 2 | 20 | 11 | 15 |
| $S_2$ | 12 | 7 | 9 | 20 | 25 |
| $S_3$ | 4 | 14 | 16 | 18 | 10 |
| Demand | 5 | 15 | 15 | 15 | |

**Solution**: $\sum s_i = \sum d_j = 50$

| Destinations / Source | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $S_1$ | 10 / 0 | 2 / 15 | 20 / 0 | 11 / 0 | 15 | 8 | 9 | | |
| $S_2$ | 12 / 0 | 7 / 0 | 9 / 15 | 20 / 10 | 25 | 2 | 2 | 11 | |
| $S_3$ | 4 / 5 | 14 / 0 | 16 / 0 | 18 / 5 | 10 | 10 | 2 | 2 | |
| Demand | 5 | 15 | 15 | 15 | | | | | |
| | 6 | 5 | 7 | 7 | | | | | |
| | | 5 | 7 | 7 | | | | | |
| | | | 7 | 2 | | | | | |

Basic variables are: S1D2=15, S2D3=15, S2D4=10, S3D1=5, S3D4=5
Non basic variables are: S1D1=S1D3=S1D4=S2D1=S2D2=S3D2=S3D3=0

The total cost of the transportation problem is:

$$Min. \quad Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} X_{ij}$$

Z= 475

***Example 2***    Find the initial basic feasible solution for the following transportation problem by VAM.

|  | | **Destination** | | | | |
|---|---|---|---|---|---|---|
|  |  | $D_1$ | $D_2$ | $D_3$ | $D_4$ | *Supply* |
| **Origin** | $O_1$ | 11 | 13 | 17 | 14 | 250 |
|  | $O_2$ | 16 | 18 | 14 | 10 | 300 |
|  | $O_3$ | 21 | 24 | 13 | 10 | 400 |
|  | *Demand* | 200 | 225 | 275 | 250 | 950 |

***Solution***   Since $\Sigma a_i = \Sigma b_j = 950$ the problem is balanced and there exists a feasible solution to the problem.

First, we find the row $\Sigma$ column penalty $P_I$ as the difference between the least and the next least cost. The maximum penalty is 5. Choose the first column arbitrarily.

In this column, choose the cell having the least cost name (1, 1). Allocate to this cell with minimum magnitude (i.e.(250, 200) = 200.) This exhausts the first column. Delete this column. Since a column is deleted, then there is a change in row penalty $P_{II}$ and column penalty $P_{II}$ remains the same. Continuing in this manner, we get the remaining allocations as given in the following table below.

I allocation

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply | $P_I$ |
|---|---|---|---|---|---|---|
| $O_1$ | 11 ⃝200 | 13 | 17 | 14 | 50 ~~250~~ | 2 |
| $O_2$ | 16 | 18 | 14 | 10 | 300 | 4 |
| $O_3$ | 21 | 24 | 13 | 10 | 400 | 3 |
| Demand | 200 / 0 | 225 | 275 | 250 | | |
| $P_I$ | 5☐ | 5 | 3 | 0 | | |

II allocation

| | $D_2$ | $D_3$ | $D_4$ | Supply | $P_{II}$ |
|---|---|---|---|---|---|
| $O_1$ | 13 ⃝50 | 17 | 14 | ~~50~~ | 2 |
| $O_2$ | 18 | 14 | 10 | 300 | 4 |
| $O_3$ | 24 | 13 | 10 | 400 | 3 |
| Demand | 225 / 175 | 275 | 250 | | |
| $P_{II}$ | 5☐ | 3 | 0 | | |

3

III allocation

| | D₂ | D₃ | D₄ | Supply | P_III |
|---|---|---|---|---|---|
| O₂ | 18 (175) | 14 | 10 | 300 125 | 4 |
| O₃ | 24 | 13 | 10 | 400 | 3 |
| Demand | 175 0 | 275 | 250 | | |
| P_III | 6 | 1 | 0 | | |

IV allocation

| | D₃ | D₄ | Supply | P_IV |
|---|---|---|---|---|
| O₂ | 14 | 10 (125) | 125 0 | 4 ← |
| O₃ | 13 | 10 | 400 | 3 |
| Demand | 275 | 250 125 | | |
| P_IV | 1 | 0 | | |

V allocation

| | D₃ | D₄ | Supply | P_V |
|---|---|---|---|---|
| O₃ | 13 (275) | 10 | 400 125 | 3 |
| Demand | 275 0 | 125 | | |
| P_V | 13 | 10 | | |

VI allocation

| | D₄ | Supply | P_VI |
|---|---|---|---|
| O₃ | 10 (125) | 125 0 | 10 ← |
| Demand | 125 0 | | |
| P_VI | 10 | | |

Finally, we arrive at the initial basic feasible solution which is shown in the following table.

| | D₁ | D₂ | D₃ | D₄ | Supply |
|---|---|---|---|---|---|
| O₁ | 11 (200) | 13 (50) | 17 | 14 | 250 |
| O₂ | 16 | 18 (175) | 14 | 10 (125) | 300 |
| O₃ | 21 | 24 | 13 (275) | 10 (125) | 400 |
| Demand | 200 | 225 | 275 | 250 | |

There are 6 positive independent allocations which equals to $m + n - 1 = 3 + 4 - 1$. This ensures that the solution is a non-degenerate basic feasible solution.

∴ The transportation cost

$$= 11 \times 200 + 13 \times 50 + 18 \times 175 + 10 \times 125$$

$$+ 13 \times 275 + 10 \times 125 = \text{Rs } 12,075.$$

Basic variables are: O1D1= 200, O1D2= 50, O2D2= 175, O2D4=125, O3D3= 275, O3D4= 125

Non basic variables are: ………………….

The total cost of the transportation problem is:

$$Min. \quad Z = \sum_{j=1}^{n}\sum_{i=1}^{m} C_{ij} X_{ij}$$

Z = 12,075

So this solution is the initial feasible solution

**H.W 7 : Solve the following Transportation Problem using:**
**1- North-west corner rule**
**2- Least cost method**
**3- VAM method**

|  | **W1** | **W2** | **W3** | **W4** | **Available** |
|---|---|---|---|---|---|
| F1 | 1 | 2 | 1 | 4 | 30 |
| F2 | 3 | 3 | 2 | 1 | 50 |
| F3 | 4 | 2 | 5 | 9 | 20 |
| Requirements | 20 | 40 | 30 | 10 | |

# Unbalanced Assignment Problem

If the cost matrix of an assignment problem is not square then the assignment problem is called an unbalanced assignment problem. In this case, dummy rows and columns are added to the matrix to form a square matrix. Then the usual assignment algorithm can be applied to this resulting balanced problem as shown in the following examples:

### A) Dummy row

| | | Machines | | | |
|---|---|---|---|---|---|
| | | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
| Jobs | $J_1$ | 20 | 25 | 22 | 28 |
| | $J_2$ | 15 | 18 | 23 | 17 |
| | $J_3$ | 19 | 17 | 21 | 24 |
| | (Dummy) | 0 | 0 | 0 | 0 |

### B) Dummy Column

| | | Machines | | | |
|---|---|---|---|---|---|
| | | $M_1$ | $M_2$ | $M_3$ | (Dummy) $M_4$ |
| Jobs | $J_1$ | 20 | 25 | 22 | 0 |
| | $J_2$ | 15 | 18 | 23 | 0 |
| | $J_3$ | 19 | 17 | 21 | 0 |
| | $J_4$ | 19 | 17 | 21 | 0 |

**Example 1:** A head department has four tasks to be performed and three subordinates. The subordinates differ in efficiency. The estimates of the time, each subordinates would take to perform is given below in the matrix. How should be allocate the tasks, one to each man, so as to minimize the total man hour?

| | | Men | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| Tasks | I | 9 | 26 | 15 |
| | II | 13 | 27 | 6 |
| | III | 35 | 20 | 15 |
| | IV | 18 | 30 | 20 |

**Solution**:

Number of Jobs $\neq$ Number of Men $\Rightarrow$ $4 \neq 3$

Dummy column is added in the matrix:

| | | Men | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Tasks | I | 9 | 26 | 15 | 0 |
| | II | 13 | 27 | 6 | 0 |
| | III | 35 | 20 | 15 | 0 |
| | IV | 18 | 30 | 20 | 0 |

Number of Jobs = Number of Men $\Rightarrow 4 = 4$
Step 1: Subtract the smallest element from each row, then subtract the smallest element from each column

| | Men | | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Tasks | I | 0 | 6 | 9 | 0 |
| | II | 4 | 7 | 0 | 0 |
| | III | 26 | 0 | 9 | 0 |
| | IV | 9 | 10 | 14 | 0 |

Now we draw the minimum number of lines :

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| I | 0 | 6 | 9 | 0 |
| II | 4 | 7 | 0 | 0 |
| III | 26 | 0 | 9 | 0 |
| IV | 9 | 10 | 14 | 0 |

Here, $N = 4 = n$ (the order of matrix).
    Then, we make the assignment.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| I | ⓪ | 6 | 9 | ⦻ |
| II | 4 | 7 | ⓪ | ⦻ |
| III | 26 | ⓪ | 9 | ⦻ |
| IV | 9 | 10 | 14 | ⓪ |

2

Job I → 1 Man = 9

Job II → 3 Man = 6

Job III → 2 Man = 20

Job IV → 4 Man = 0

Total cost is Z= $\displaystyle\sum_{i=1}^{4}\sum_{j=1}^{4}c_{ij}x_{ij}$  ⇒ Z = 35

**Example 2:** A company has 4 machines to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given as follows. Determine the job assignments which will minimize the total cost.

| Machine<br>Job | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 18 | 24 | 28 | 32 |
| B | 8 | 13 | 17 | 18 |
| C | 10 | 15 | 19 | 22 |

*Solution* Since the cost matrix is not a square matrix we add a dummy row *D* with all the elements 0.

| Machine<br>Job | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 18 | 24 | 28 | 32 |
| B | 8 | 13 | 17 | 18 |
| C | 10 | 15 | 19 | 22 |
| D | 0 | 0 | 0 | 0 |

Subtract the minimum element in each row from all the elements in its row.

| Machine<br>Job | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 0 | 6 | 10 | 14 |
| B | 0 | 5 | 9 | 10 |
| C | 0 | 5 | 9 | 12 |
| D | 0 | 0 | 0 | 0 |

3

Since each column has minimum element 0, we draw minimum number of lines to cover all zeroes.

| Machine / Job | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 0 | 6 | 10 | 14 |
| B | 0 | 5 | 9 | 10 |
| C | 0 | 5 | 9 | 12 |
| D | 0 | 0 | 0 | 0 |

∵the number of lines drawn to cover all zeroes = 2 < the order of matrix, we form a second modified matrix.

| Machine / Job | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 0 | 1 | 5 | 9 |
| B | 0 | 0 | 4 | 5 |
| C | 0 | 0 | 4 | 7 |
| D | 5 | 0 | 0 | 0 |

Here, $N = 3 < n = 4$. Again, we subtract the smallest uncovered element from all the uncovered elements and add to the element at the point of intersection.

| Machine / Job | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 0 | 1 | 1 | 4 |
| B | 0 | 0 | 0 | 1 |
| C | 0 | 0 | 0 | 3 |
| D | 9 | 4 | 0 | 0 |

Here, $N = 4 = n$. Hence, we make an assignment.

| Machine Job | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | ⓪ | 1 | 1 | 4 |
| B | X̸ | ⓪ | X̸ | 1 |
| C | X̸ | X̸ | ⓪ | 3 |
| D | 9 | 4 | X̸ | ⓪ |

Job A → Machine 1 = 18

Job B → Machine 2 = 13

Job C → Machine 3 = 19

Job D → Machine 4 = 0    Since $D$ is a dummy job, machine 4 has assigned no job.

Total cost is Z= $\sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}x_{ij}$  $\Rightarrow Z = 50$

5

)

**TRANSSHIPMENT PROBLEMS**

Sometimes a point in the shipment process can both receive goods from other points and send goods to other points. This point is called as **transshipment point** through which goods can be transshipped on their journey from a supply point to demand point.

Shipping problem with this characteristic is a transshipment problem.

The optimal solution to a transshipment problem can be found by converting this transshipment problem to a transportation problem and then solving this transportation problem.

### *Remark*

As stated in "Formulating Transportation Problems", we define a **supply point** to be a point that can send goods to another point but cannot receive goods from any other point.

Similarly, a **demand point** is a point that can receive goods from other points but cannot send goods to any other point.

### Steps

1. If the problem is unbalanced, balance it

   Let $s$ = total available supply (or demand) for balanced problem

2. Construct a transportation tableau as follows

   A row in the tableau will be needed for each supply point and transshipment point

   A column will be needed for each demand point and transshipment point

   Each supply point will have a supply equal to its original supply

   Each demand point will have a demand equal to its original demand

   Each transshipment point will have a supply equal to "that point's original supply + $s$"

   Each transshipment point will have a demand equal to "that point's original demand + $s$"

3. Solve the transportation problem

### Example

Bosphorus manufactures LCD TVs at two factories, one in Istanbul and one in Bruges. The Istanbul factory can produce up to 150 TVs per day, and the Bruges factory can produce up to 200 TVs per day. TVs are shipped by air to customers in London and Paris. The customers in each city require 130 TVs per day. Because of the deregulation of air fares, Bosphorus believes that it may be cheaper to first fly some TVs to Amsterdam or Munchen and then fly them to their final destinations.

The costs of flying a TV are shown at the table below. Bosphorus wants to minimize the total cost of shipping the required TVs to its customers.

| € | To | | | | | |
|---|---|---|---|---|---|---|
| From | Istanbul | Bruges | Amsterdam | Munchen | London | Paris |
| Istanbul | 0 | - | 8 | 13 | 25 | 28 |
| Bruges | - | 0 | 15 | 12 | 26 | 25 |
| Amsterdam | - | - | 0 | 6 | 16 | 17 |
| Munchen | - | - | 6 | 0 | 14 | 16 |
| London | - | - | - | - | 0 | - |
| Paris | - | - | - | - | - | 0 |

**Answer:**

In this problem Amsterdam and Munchen are *transshipment points*.

**Step 1.** Balancing the problem

Total supply = 150 + 200 = 350

Total demand = 130 + 130 = 260

Dummy's demand = 350 – 260 = 90

$s$ = 350 (total available supply or demand for balanced problem)

**Step 2.** Constructing a transportation tableau

Transshipment point's demand = Its original demand + $s$ = 0 + 350 = 350

Transshipment point's supply = Its original supply + $s$ = 0 + 350 = 350

| | Amsterdam | Munchen | London | Paris | Dummy | Supply |
|---|---|---|---|---|---|---|
| Istanbul | 8 | 13 | 25 | 28 | 0 | 150 |
| Bruges | 15 | 12 | 26 | 25 | 0 | 200 |
| Amsterdam | 0 | 6 | 16 | 17 | 0 | 350 |
| Munchen | 6 | 0 | 14 | 16 | 0 | 350 |
| Demand | 350 | 350 | 130 | 130 | 90 | |

**Step 3.** Solving the transportation problem

| | Amsterdam | Munchen | London | Paris | Dummy | Supply |
|---|---|---|---|---|---|---|
| Istanbul | 8 / 130 | 13 | 25 | 28 | 0 / 20 | 150 |
| Bruges | 15 | 12 | 26 | 25 / 130 | 0 / 70 | 200 |
| Amsterdam | 0 / 220 | 6 | 16 / 130 | 17 | 0 | 350 |
| Munchen | 6 | 0 / 350 | 14 | 16 | 0 | 350 |
| Demand | 350 | 350 | 130 | 130 | 90 | 1050 |

**Report:**

Bosphorus should produce 130 TVs at Istanbul, ship them to Amsterdam, and transship them from Amsterdam to London.

The 130 TVs produced at Bruges should be shipped directly to Paris.

The total shipment is 6370 Euros.

## Example

Use the simplex method to solve the (LP) model:

$$min\ Z = -6x_1 - 10x_2 - 4x_3$$

Subject to

$$x_1 + x_2 + x_3 \leq 1000$$

$$x_1 + x_2 \qquad \leq 500$$

$$x_1 + 2x_2 \qquad \leq 700$$

$$x_1, x_2, x_3 \qquad \geq 0$$

<u>**Solution:**</u>

$$min \; Z + 6x_1 + 10x_2 + 4x_3 = 0$$

**Subject to**

$$x_1 + x_2 + x_3 + S_1 = 1000$$

$$x_1 + x_2 \quad + S_2 \qquad = 500$$

$$x_1 + 2x_2 \quad + S_3 \quad = 700$$

$$x_1, x_2, x_3 , S_1, S_2 \; S_3 \; \geq 0$$

<u>**Table 1:**</u>

| Basic | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | Sol. |
|-------|-------|-------|-------|-------|-------|-------|------|
| $S_1$ | 1 | 1 | 1 | 1 | 0 | 0 | 1000 |
| $S_2$ | 1 | 1 | 0 | 0 | 1 | 0 | 500 |
| $S_3$ | 1 | 2 | 0 | 0 | 1 | 1 | 700 |
| Max Z | 6 | 10 | 4 | 0 | 0 | 0 | 0 |

$\frac{1000}{1} = 1000$

$\frac{500}{1} = 500$

$\frac{700}{2} = \boxed{350}$

**New $S_3$-row or $x_2$-row** $= \frac{1}{2}$[1  2  0   0  0  1   700]

$\qquad\qquad\qquad\qquad = [\frac{1}{2}$  1  0  0  0  $\frac{1}{2}$   350]

**New $S_1$-row** = [1  1  1  1  0  0   1000]-(1)[$\frac{1}{2}$  1  0  0  0  $\frac{1}{2}$   350]

$\qquad\qquad = [\frac{1}{2}$  0  1  1  0  $-\frac{1}{2}$   650]

**New $S_2$-row** = [1  1  0  0  1  0   500]-(1)[$\frac{1}{2}$  1  0  0  0  $\frac{1}{2}$   350]

$\qquad\qquad = [\frac{1}{2}$  0  0  0  1  $-\frac{1}{2}$   150]

**New $Z$-row** = [6  10  4  0  0  0   0]-(10)[$\frac{1}{2}$  1 0  0  0  $\frac{1}{2}$   350]

$\qquad\qquad$ =[1   0   4   0   0  - 5    -3500]

## Table 2:

| Basic | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | Sol. |
|-------|-------|-------|-------|-------|-------|-------|------|
| $S_1$ | 1/2 | 0 | 1 | 1 | 0 | -1/2 | 650 |
| $S_2$ | 1/2 | 0 | 0 | 0 | 1 | -1/2 | 150 |
| $x_2$ | 1/2 | 1 | 0 | 0 | 0 | 1/2 | 350 |
| Max Z | 1 | 0 | 4 | 0 | 0 | -5 | -3500 |

$\frac{650}{1} = $ 650

$\frac{150}{0} = \infty$ $\qquad$ **(ignore)**

$\frac{350}{0} = \infty$ $\qquad$ **(ignore)**

**New $S_1$-row or $x_3$-row** =1[$\frac{1}{2}$  0  1   1  0  -$\frac{1}{2}$  650]

$\qquad\qquad$ =[$\frac{1}{2}$  0  1   1  0  -$\frac{1}{2}$  650]

**New $S_2$-row** = [$\frac{1}{2}$  0  0  0  1 -$\frac{1}{2}$  150]-(0)[$\frac{1}{2}$  0  1   1  0  -$\frac{1}{2}$  650]

$\qquad\qquad$ =[$\frac{1}{2}$  0  0  0  1 -$\frac{1}{2}$  150]

**New $x_2$-row** = [$\frac{1}{2}$  1  0  0  0  $\frac{1}{2}$  350]-(0)[$\frac{1}{2}$  0  1   1  0  -$\frac{1}{2}$  650]

$\qquad\qquad$ =[$\frac{1}{2}$  1  0  0  0  $\frac{1}{2}$  350]

**New $Z$-row** = [1  0  4  0  0  -5   -3500]-(4)[$\frac{1}{2}$  0  1   1  0  -$\frac{1}{2}$  650]

$\qquad\qquad$ =[-1  0  0  -4  0   -3   -6100]

## Table 3: (optimal solution):

| Basic | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | Sol. |
|-------|-------|-------|-------|-------|-------|-------|------|
| $x_3$ | 1/2 | 0 | 1 | 1 | 0 | -1/2 | 650 |
| $S_2$ | 1/2 | 0 | 0 | 0 | 1 | -1/2 | 150 |
| $x_2$ | 1/2 | 1 | 0 | 0 | 0 | 1/2 | 350 |
| Max Z | -1 | 0 | 0 | -4 | 0 | -3 | -6100 |

## The optimal solution :

$x_3 = 650$ , $x_2 = 350$ , $S_1 = 0$  $S_3$ & $S_2 = 150$  $x_1 = 0$ // Z=280

# Definition of the General LP Problem

■ **Definition**: a function $f(x_1, x_2, \ldots, x_n)$ of the $n$ variables $x_1, x_2, \ldots, x_n$ is a $\boxed{\textbf{linear function}}$ if and only if there is a set of <u>constants</u> $c_1, c_2, \ldots, c_n$ such that

$$
\begin{aligned}
f(x_1, x_2, \ldots, x_n) &= c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
&= \underline{c}^T \underline{x} \text{ , say.}
\end{aligned}
$$

■ **Definition**: for any <u>linear function</u> $f(x_1, x_2, \ldots, x_n)$ and any real value $b$, a $\boxed{\textbf{linear inequality}}$ is a relation taking one or other of the following forms:

$$
f(x_1, x_2, \ldots, x_n) \leq b
$$

or

$$
f(x_1, x_2, \ldots, x_n) \geq b \quad .
$$

■ **Definition**: for any <u>linear function</u> $f(x_1, x_2, \ldots, x_n)$ and any real value $b$, a $\boxed{\textbf{linear equality}}$ is a relation of the following form:

$$
f(x_1, x_2, \ldots, x_n) = b \quad .
$$

■ **Definition**: A | **Linear Programming Problem** | is an optimisation problem
in which

- we are required to **optimise** (that is, <u>maximise</u> or <u>minimise</u>) a **linear function** of the decision variables;

- the values of the decision variables are required to satisfy a set of **linear constraints** (that is, <u>linear inequalities</u> and / or <u>linear equalities</u>);

- a **sign restriction** is associated with each decision variable (that is, each decision variable $x_i$ is required to be either <u>non-negative</u> or <u>unrestricted</u>).

# Standard Form of the General LP Problem

■ **Definition**: A Linear Programming Problem is in **Standard Form** if it is written in the following way:

$$\max \ \underline{c}^T \underline{x} \ ,$$

$$\text{s.t. } A \ \underline{x} = \underline{b} \ ,$$

$$\underline{x} \geq \underline{0} \ ,$$

where

- $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ ,

- $\underline{c} = (c_1, c_2, \dots, c_n)^T$ , $\underline{x} = (x_1, x_2, \dots, x_n)^T$ , $\underline{b} = (b_1, b_2, \dots, b_m)^T$ ,

- $b_i \geq 0 \ \forall \, i \in [1, m]$ , $n > m$ and $rank(A) = m$ .

**Note** that $A$ always has <u>more columns than rows</u>. This means that the system $A \ \underline{x} = \underline{b}$ has **more variables than equations** (so, <u>usually</u>, there are <u>many solutions</u> to the system).

At this stage, it is very important to be able to show that <u>any</u> LP problem can be transformed into the **Standard Form**. We do this by showing how to deal with "situations" that don't (at first sight) seem to fit into the **Standard Form**:

1. If a constraint is an **equality** with a <u>negative</u> right-hand-side (RHS), then we multiply it by $-1$. For example:

$$x_1 - x_2 = -2 \quad \Rightarrow \quad -x_1 + x_2 = 2$$

2. If an **inequality constraint** has a <u>negative</u> RHS, multiply it by $-1$ (which <u>reverses</u> the direction of the inequality) and then apply Step 3 to it (see below). For example:

$$2x_1 - x_2 \geq -2 \quad \Rightarrow \quad -2x_1 + x_2 \leq 2$$

3. For any **inequality constraint** with non-negative RHS, include a **slack** variable for a $\leq$ constraint or a **surplus** variable for a $\geq$ constraint. For example:

$$-2x_1 + x_2 \leq 2 \quad \Rightarrow \quad -2x_1 + x_2 + s_1 = 2 \text{ and } s_1 \geq 0 \ ;$$

$$-2x_1 + x_2 \geq 2 \quad \Rightarrow \quad -2x_1 + x_2 - s_2 = 2 \text{ and } s_2 \geq 0 \ .$$

4. If a particular variable is <u>unrestricted</u> in sign, we re-express it as the difference of two **sign-restricted** variables. For example:

$$x_i \text{ unrestricted} \quad \Rightarrow \quad x_i = s_3 - s_4 \text{ with } s_3, s_4 \geq 0 \ .$$

5. If we are given a **minimisation** problem, we just multiply the objective function by $-1$ and then **maximise**. E.g.:

$$\boxed{\min\ (4x_1 - 5x_2)} \quad \text{is equivalent to} \quad \boxed{\max\ (-4x_1 + 5x_2)}$$

**Example**

If we are given the problem

$$\min \ -3x_1 - 2x_2$$

s.t.

$$
\begin{array}{rcl}
x_1 - 2x_2 & = & -2 \\
2x_1 + x_2 & \leq & 6 \\
-2x_1 - 3x_2 & \leq & -14 \\
x_1, \ x_2 & \geq & 0
\end{array}
$$

$$\updownarrow$$

$$
\begin{array}{llllllllcl}
\max & 3x_1 & + & 2x_2 & + & 0s_1 & + & 0s_2 & & \\
\text{s.t.} & -x_1 & + & 2x_2 & & & & & = & 2 \\
& 2x_1 & + & x_2 & + & s_1 & & & = & 6 \\
& 2x_1 & + & 3x_2 & - & s_2 & & & = & 14 \\
& x_1 & , & x_2 & , & s_1 & , & s_2 & \geq & 0
\end{array}
$$

**Note**: $s_1$ is a <u>slack</u> variable and $s_2$ is a <u>surplus</u> variable.

**Another example:**

We have seen that this problem can be written in the form:

$$\max \ 5000E + 4000F$$

$$
\begin{aligned}
\text{s.t.} \quad 10E + 15F &\leq 150 \\
20E + 10F &\leq 160 \\
30E + 10F &\geq 135 \\
E - 3F &\leq 0 \\
E + F &\geq 5 \\
E, F &\geq 0 \ .
\end{aligned}
$$

The **Standard Form** for this problem is:

$$\max \ 5000E + 4000F + 0s_1 + 0s_2 + 0s_3 + 0s_4 + 0s_5$$

s.t.

$$
\begin{aligned}
10E \ +15F \ +s_1 &&&&&= 150 \\
20E \ +10F \ &+s_2 &&&&= 160 \\
30E \ +10F \ &&-s_3 &&&= 135 \\
E \ -3F \ &&&+s_4 &&= 0 \\
E \ +F \ &&&&-s_5 &= 5
\end{aligned}
$$

$$E, F, s_1, s_2, s_3, s_4, s_5 \ \geq \ 0 \ .$$

## Other Types of OR Problem

To conclude this first section, we briefly consider two types of OR problem which have some similarities to LP problems but, also, important <u>differences</u>:

1. **Integer Programming**

2. **Nonlinear Programming**

### Integer Programming

Consider the following problem:

$$\max \quad z = 5x_1 + 27x_2$$

s.t.

$$
\begin{aligned}
2x_1 + 11x_2 &\leq 59 \\
x_1 - x_2 &\leq 7 \\
x_1, \ x_2 &\geq 0 \ .
\end{aligned}
$$

The solution is found to be: $x_1^* = 10\frac{6}{13}, \ x_2^* = 3\frac{6}{13}, \ z^* = 145\frac{10}{13}$.

Now suppose that we include the additional constraints that $x_1$ and $x_2$ must be **integers** (recall that the MachineCo problem considered earlier, if it is to be genuinely realistic, should include similar constraints). Such extra constraints change the nature of the problem considerably. How do we solve such a problem? We could try <u>rounding</u> the solution of the original problem:

| $x_1$ | $x_2$ | $z$ | feasible? |
|:---:|:---:|:---:|:---:|
| 10 | 3 | 131 | $\checkmark$ |
| 10 | 4 | 158 | $\times$ |
| 11 | 3 | 136 | $\times$ |
| 11 | 4 | 163 | $\times$ |

However, the <u>actual</u> solution of the problem with the integer constraints is

$$x_1^* = 2, \ x_2^* = 5, \ z^* = 145 \ .$$

This point does not correspond to any of the "rounded" versions considered above, so <u>rounding</u> is not a reliable technique for solving **Integer Programming** problems.