Nonlinear Programming

Consider the following problem:

$$\max \quad z = 60x_1 + 80x_2 - 5x_1^2 - 4x_2^2$$

s.t.

$$\begin{aligned} 6x_1 + 5x_2 &\leq 60 \\ x_1 + 2x_2 &\leq 15 \\ x_1 \quad\quad &\leq 8 \\ x_1,\ x_2 &\geq 0\,. \end{aligned}$$

(Note that there are **nonlinear** terms involved here. In this case, they only appear in the objective function but they <u>could</u> appear in the constraints also. To make matters worse, the nonlinear terms could include <u>higher</u> powers and also <u>functions</u> such as “cos”, “exp”, “log”, etc., etc.)

The solution of the given problem is found to be:
$x_1^* = 4\frac{1}{6},\ x_2^* = 5\frac{5}{12},\ z^* = 479\frac{1}{6}$. Drawing a graphical representation of this problem shows clearly that the optimal point does **not** occur at a <u>corner point</u> of the feasible region (unlike in LP problems). Therefore, the kind of methods we are going to develop for solving LP problems (methods which involve examining corner points of the feasible region) **will not work** for **Nonlinear Programming** problems.

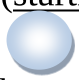
Network analysis

Many important optimization problems can best be analyzed using graphical or network representation. Network models can be used as an aid in the scheduling of large complex projects that consist of many activities. If the duration of each activity is known with certainty, the **Critical Path Method (CPM)** can be used to determine the length of the time required to complete a project. CPM also can be used to find how long each activity in the project can be delayed without delaying the compilation of the project. CPM was developed in the late 1950s by researchers at du Pont and Sperry Rand.

If the duration of the activity is not known with certainty, the **Program Evaluation and Review Technique (PERT)** can be used to estimate the probability that the project will be complete by the given deadline. PERT was developed in the late 1950s by consultants working on the development of the Polaris missile.
The basic difference between them is that CPM is a deterministic technique while PERT is a probabilistic technique.
PERT and CPM are the two most popular network analysis technique used in many applications, including; Scheduling construction projects, Building a ship, Designing and marketing a new product, and Installing a new computer.

Basic Network Terminology:

A graph, or **network ,** is defined by two sets of symbols : node (event) and arcs (arrow (→)) (activities). It is a representation of all the activities, events, dummy activities.

Event: We refer to the nodes in our project as events (<u>starting or completion of a job</u>). It is represented in a network by a circle (O). 
Activity: We need a list of the activities that make up the **project**. This consumes resources (Time, human resources, money, and material). The project is considered to be complete when all the activities have been complete.

For each activity, there is a set of activities called (**predecessors** of the activity) that must be completed before the activity begins.

Dummy Activities: This is an activity that does not take time or resources. It is used merely to show logical dependences or sequences between activities so as not to violate the rules of the drawing network. It is represented in a network by a dotted arrow (- →).

Given a list of activities and predecessors, here the type of **project network** is called **AOA (activity on the arc)**. AOA is constructed by using some rules:

1- Node 1 represents the start of the project. An arc should lead from node 1 to present each activity that has no predecessors.

2- A node (called the finish node) representing the completion of the project which should be included in the network.

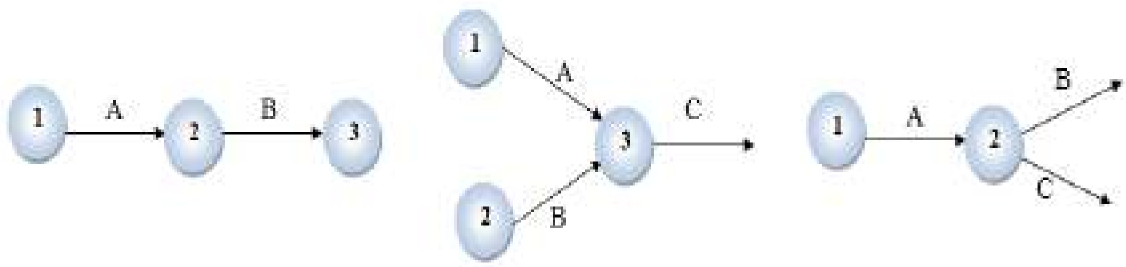
3- Number of nodes in the network so that the node representing the completion of activity always has a large number than the one representing the beginning of an activity.

4- An activity should not be represented by more than one arc.

5- Two nodes can be connected by at most one arc.

6- Loops are not allowed i.e. series of activities that leads back to the same event.

Example 1: The diagrams represent a project with three activities **A, B, C**.
1- Activity **A** must be completed before activity **B** can begin.
2- Activity **A** and **B** must be completed before activity **C** can begin.
3- Activity **A** must be completed before activity **B** and **C** can begin.



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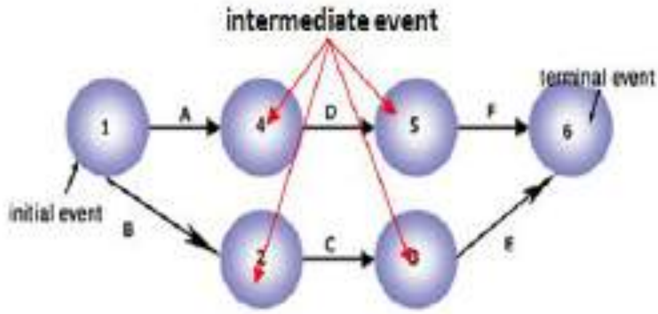

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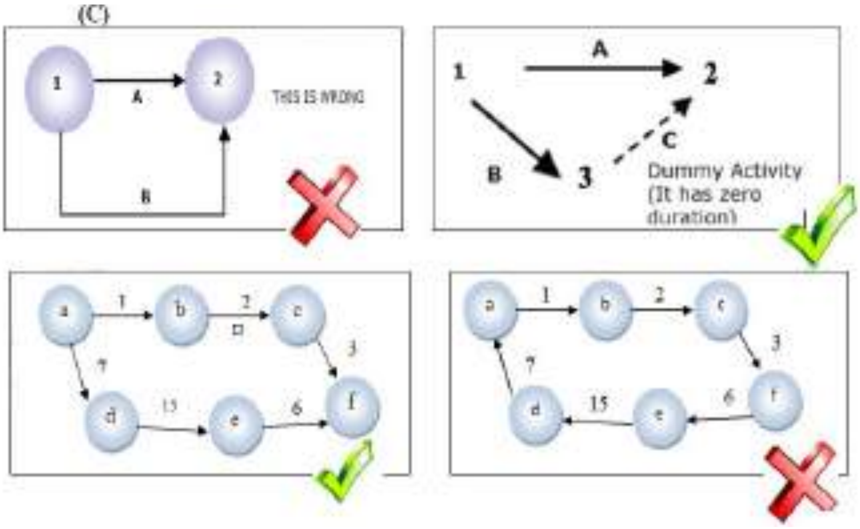
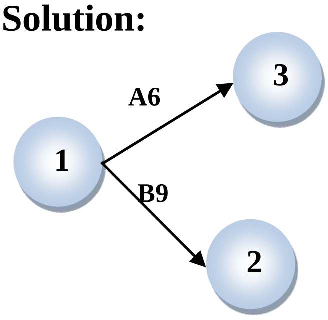
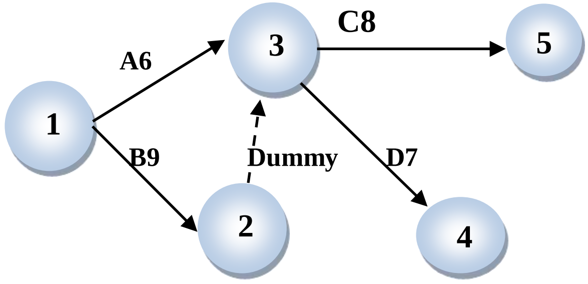


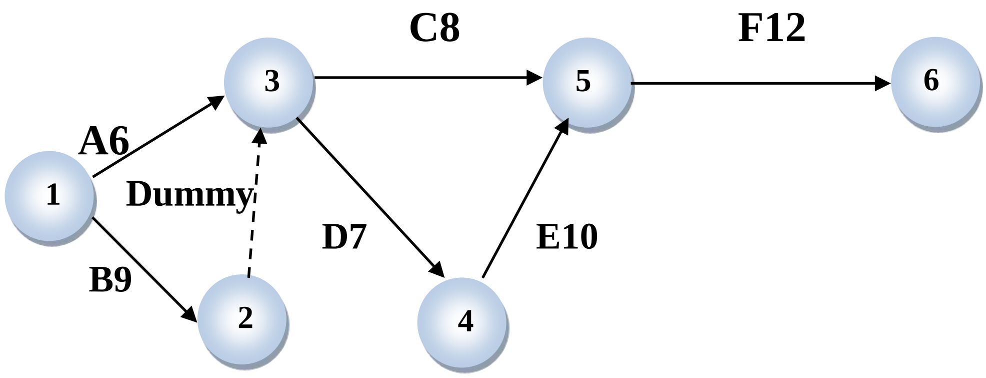
Example (2): Draw the diagram for the following activities

Activities	Predecessors	Duration (Days)
A	-----	6
B	-----	9
C	A, B	8
D	A, B	7
E	D	10
F	C,E	12

Solution:



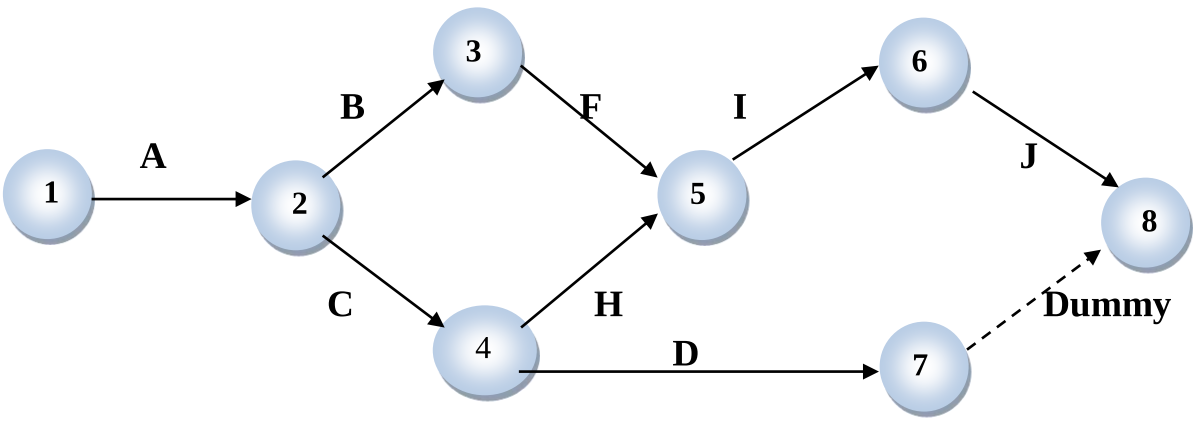




Example 3: Draw the diagram for the following activities

Activities	**Predecessors**
A	-----
B	A
C	A
D	C
F	B
H	C
I	F, H
J	I

Solution: Finish the graph



Critical Path method (CPM)

Assuming the duration of each activity is known, the critical path method may be used to find the duration of the project which is the longest path in a project network. It can be calculated using the Inspection/Tree diagram method or Forward/Backward pass method.

- **Critical activities,** if it starts and finishes times are predetermined (fixed). A delay in the start time causes a delay in the completion time.
-
- **Noncritical activity**, if it can be scheduled in a time greater than its duration.

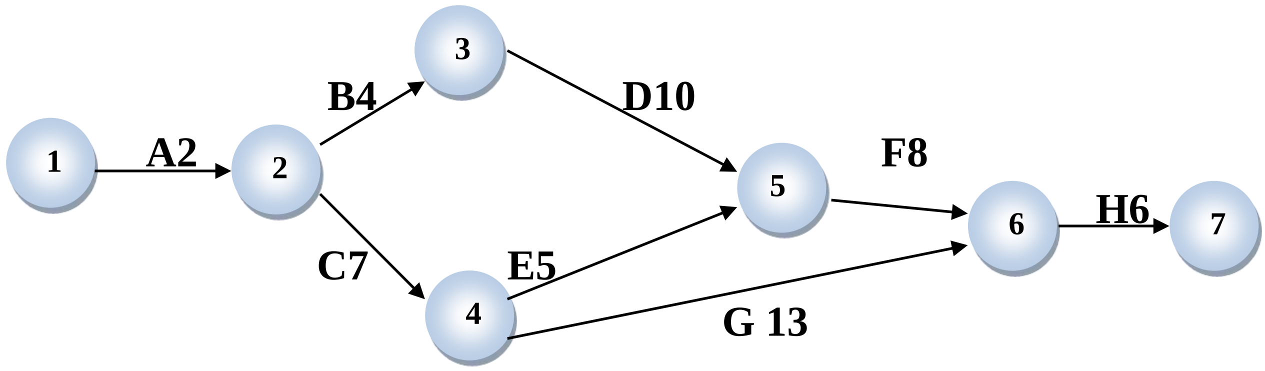
Determination of Critical Path

Step 1: List all the possible sequences from start to finish.

Step 2: For each sequence determines the total time required from start to finish.

Step 3: Identify the longest path (Critical Path)

Example 1: Use the Inspection /Tree diagram method to calculate the critical path of the following diagram.



Solution:
Step1. List all the possible sequences from start to finish

Path 1: 1 → 2 → 3 → 5 → 6→ 7

Path 2: $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7$

Path 3: $1 \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow 7$

Step2. For each sequence determine the total time required from start to finish.

Path1: $\Rightarrow$ 30 days

Path2: $\Rightarrow$ 28 days

Path3: $\Rightarrow$ 28 days

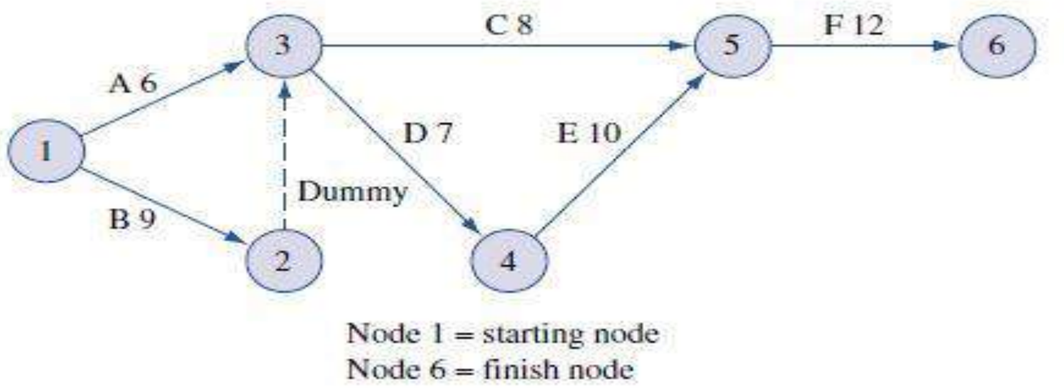
Step3. Identify the longest path (Critical Path)

Path1: $1 + 2 + 3 + 5 + 6 + 7 = 30$ days

Example 2: A company is about to introduce a new product (product 3). One unit of product 3 is produced by assembling 1 unit of product 1 and 1 unit of product 2. Before production begins on either product 1 or 2, raw materials must be purchased and workers must be trained. Before products 1 and 2 can be assembled into product 3, the finished product 2 must be inspected. A list of activities and their predecessors and of the duration of each activity is given in theTable below. Draw a project diagram for this project and find the critical path.

Activities (Jobs)	**Activities Predecessors**	**Duration Days**
A	-	6 Days
B	-	9 Days
C	A, B	8 Days
D	A, B	7 Days
E	D	10 Days
F	C,E	12 Days

Solution:



Step1. List all the possible sequences from start to finish

Path 1: 1 → 3 → 5 → 6

Path 2: 1 → 2 → 3→ 4 → 5 → 6

Step2. For each sequence determine the total time required from start to finish.

Path1: ⇒ 26 days

Path2: ⇒ 38 days

Step3. Identify the longest path (Critical Path)

Path1: 1 + 2 + 3 + 4 + 5 + 6 = 38 days

H.W: Construct an arrow diagram for the following **job project and** find the critical path :

Activities (Jobs)	**Activities (Jobs) Predecessors**	**Duration Days**
A	-	4 Days
B	-	5 Days
C	A,B	10 Days
D	A	7 Days
E	D	8 Days
F	C,E	6 Days

Forward/Backward Pass Method

The CPM has two concepts of **Early Event Time (ET)** which **forward pass** that determines the earliest occurrence times of the events and **Late Event Time (LT)** which is a **backward pass** for an event that calculates their latest occurrence times.

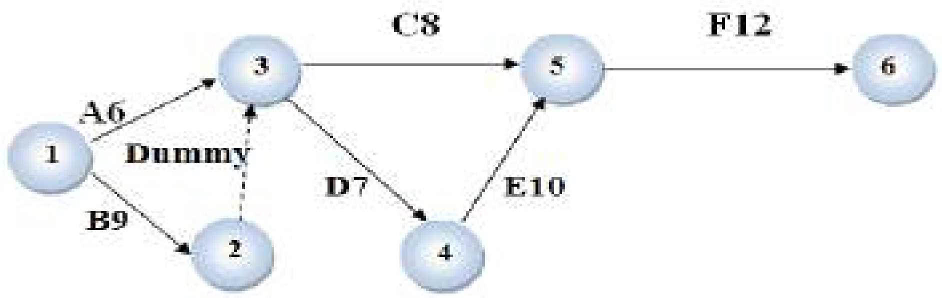
Earliest Start Time :

Definition1: The early event time (forward pass) for node *i*, represented ET(*i*), is the earliest time at which the event corresponding to node *i* can occur.
We compute ET(*i*) as follows:

1- Find each prior event (**immediate predecessor**) to node *i* that is connected by an arc to node *i*.
2- To the ET for each immediate predecessor of node *i*, add the duration of the activity connecting the immediate predecessor to node *i*. Note that, the first event of the diagram has no prior events going into it. So set **Et(1)=0.**

3- ET(*i*) equals the maximum of the sums computed in step 2.

Example1: For example 2 in the previous lecture compute the ET(*i*)'s.



Solution:
ET(1) = 0

Node 1 is the only immediate predecessor of node 2, so:
ET(2) = ET(1) + 9 = 9

The immediate predecessors of node 3 are nodes 1 and 2. Thus:

$$ET(3) = \max\begin{cases} ET(1) + 6 = 0 + 6 = 6 \\ ET(2) + 0 = 9 + 0 = 9 \end{cases} = 9$$

Node 4's only immediate predecessor is node 3. Thus,
ET(4) = ET(3) + 7 = 9 + 7 = 16

Node 5's immediate predecessors are nodes 3 and 4. Thus,

$$ET(5) = \max\begin{cases} ET(3) \ + 8 \ = 9 \ + 8 \ = \ 17 \\ ET(4) + 10 = 16 + 10 = 26 \end{cases} = 26$$

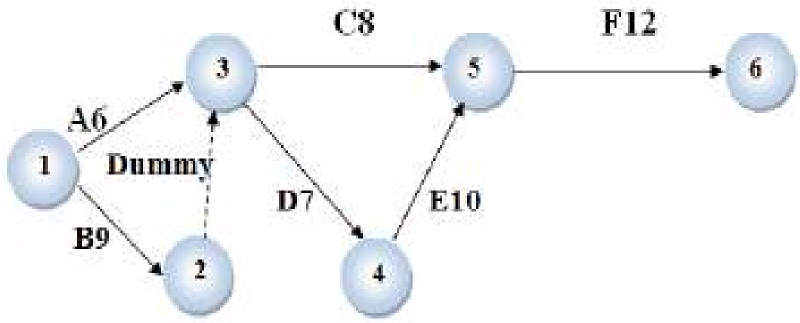
Finally, node 5 is the only immediate predecessor of node 6. Thus,
ET(6) = ET(5) + 12 = 26 + 12 = 38.

Because node 6 represents the completion of the project, we see that the earliest time that product 3 can be assembled is 38 days from now.
It can be shown that $ET(i)$ is the length of the longest path in the project network from node 1 to node *i*.

Latest Completion Time

Definition2: The late event time (backward pass) for node *i*, represented LT(*i*), is the latest time at which the event corresponding to node *i* can occur without delaying the completion of the project.
We compute LT(*i*) as follows:

1- Find each node that occurs after node *i* and is connected to node *i* by an arc. These events are the (**immediate predecessors**) of node *i*.
2- From the LT for each immediate predecessor of node *i*, subtract the duration of the activity joining successor to node *i*.
3- LT(*i*) is the smallest of the differences computed in step 2.



Find the LT(*i*)'s for the example ?

Solution:

$LT(6) = 38$

Because node 6 is the only immediate successor of node 5,
$LT(5) = LT(6) - 12 = 38 - 12 = 26$

Node 4's only immediate successor is node 5. Thus,
$LT(4) = LT(5) - 10 = 26 - 10 = 16$

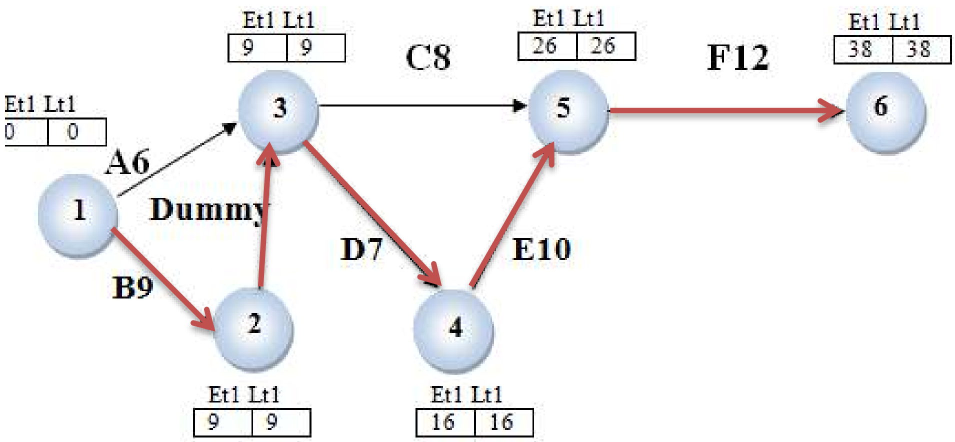
Nodes 4 and 5 are immediate successors of node 3. Thus,

$$LT(3) = \min\begin{cases} LT(4) - 7 = 16 - 7 = 9 \\ LT(5) - 8 = 26 - 8 = 18 \end{cases} = 9$$

Node 3 is the only immediate successor of node 2. Thus,
LT(2) = LT(3) - 0 = 9

Finally, node 1 has nodes 2 and 3 as immediate successors. Thus,

$$LT(1) = \min\begin{cases} LT(3) - 6 = 9 - 6 = 3 \\ LT(2) - 9 = 9 - 9 = 0 \end{cases} = 0$$



Note:

1- If LT(*i*) = ET(*i*) , any delay in the occurrence of node *i* will delay the completion of the project. For example, since LT(4) = ET(4), any delay in the occurrence of node 4 will delay the completion of the project.

2-Based on the preceding calculations, an activity (i, j) will be critical if LT(*i*) = ET(*i*)

Float Time

Before the project is begun, the duration of an activity is unknown, and the duration of each activity used to construct the project network is just an estimate of the activity's actual completion time. The concept of total float of an activity can be used as a measure of how important it is to keep each activity's duration from greatly exceeding our estimate of its completion time.
Spare time is known as float time and is important because it may be useful to know how much time a task can be delayed without having an impact on the whole process. Float refers to the amount of time by which a particular event or an activity can be delayed without affecting the time schedule of the network.

There are three main:

1- Total Float
2- Free Float
3- Independent Float

Total Float (TF): Total float of an activity is the amount by which the duration of the activity can be increased without delaying the completion of the project.

Let t_{ij} be the duration of activity (i, j), then *TF*(i, j) can be expressed in terms LT(j) and ET(i). Activity (i, j) begins at node i. The Total Float is calculated by the following formula:

$$\boldsymbol{TF(i,j) = LT(j) - ET(i) - t_{ij}}$$

Find the $\boldsymbol{TF(i,j)}$ for the example above?

Activity B: $TF(1,2) = LT\ (2) - ET(1) - 9 = 0$

Activity A: $TF(1,3) = LT\ (3) - ET(1) - 6 = 3$

Activity D: $TF(3,4) = LT\ (4) - ET(3) - 7 = 0$

Activity C: $TF(3,5) = LT\ (5) - ET(3) - 8 = 9$

Activity E: $TF(4,5) = LT\ (5) - ET(4) - 10 = 0$

Activity F: $TF(5,6) = LT\ (6) - ET(5) - 12 = 0$

Dummy Activity: $TF(2,3) = LT\ (3) - ET(2) - 0 = 0$

Note: If an activity has a **total float of zero**, any delay in the start of the activity (or duration of the activity) will delay the completion of the project.

Such activity is called **critical activity**. In fact, increasing the duration of an activity by Δ days will increase the length of the project by Δ days.

Free Float (FF):
As we have seen, the total float of an activity can be used as a measure of the flexibility in the duration of an activity. For example, activity A can take up to 3 days longer than its scheduled duration of 6 days without delaying the completion of the project. Another measure of the flexibility available in the duration of an activity is free float which is calculated by the following formula:

$$\boldsymbol{FF(i,j) = ET(j) - ET(i) - t_{ij}}$$

Free Float of the activity corresponding to the arc (i, j), denoted by FF(i, j), is the amount by which the starting time of the activity corresponding to arc (i, j) (or the duration of the activity) that can be delayed without delaying the start of any later activity beyond its earliest possible starting time.

Find the $\boldsymbol{FF(i,j)}$ for the example above?

Activity B: $FF(1,2) = ET(2) - ET(1) - t_{12} = 9 - 0 - 9 = 0$

Activity A: $FF(1,3) = 9 - 0 - 6 = 3$

Activity D: $FF(3,4) = 16 - 9 - 7 = 0$

Activity C: $FF(3,5) = 26 - 16 - 8 = 9$

Activity E: $FF(4,5) = 26 - 16 - 10 = 0$

Activity F: $FF(5,6) = 38 - 26 - 12 = 0$

For example, because the free float for activity C is 9 days, a delay in the start of activity C (or in the occurrence of node 3) or a delay in the duration of activity C of more than 9 days will delay the start of some later activity (in this case, activity F).

Note: Critical Path is the longest sequence of activity on a project that carries zero free float

Independent Float (IF) :
Independent Float is the amount of time that can be delayed without affecting either predecessor or successor activities. And denoted by (IF). The Independent Float is calculated by the following formula:

Independent Float for activity (i, j)

$$IF(i,j) = ET(j) - LT(i) - t_{ij}$$

H.W: With the help of the activities given in the table draw a network. Determine its earliest occurrence time, latest occurrence time, critical path, TF, FF and IF.

Activity	Duration
1–2	4 days
1–3	12 days
1–4	10 days
2–4	8 days
2–5	6 days
3–6	8 days
4–6	10 days
5–7	10 days
6–7	0 days
6–8	8 days
7–8	10 days
8–9	6 days

Project Evaluation and Review Technique (PERT) Method

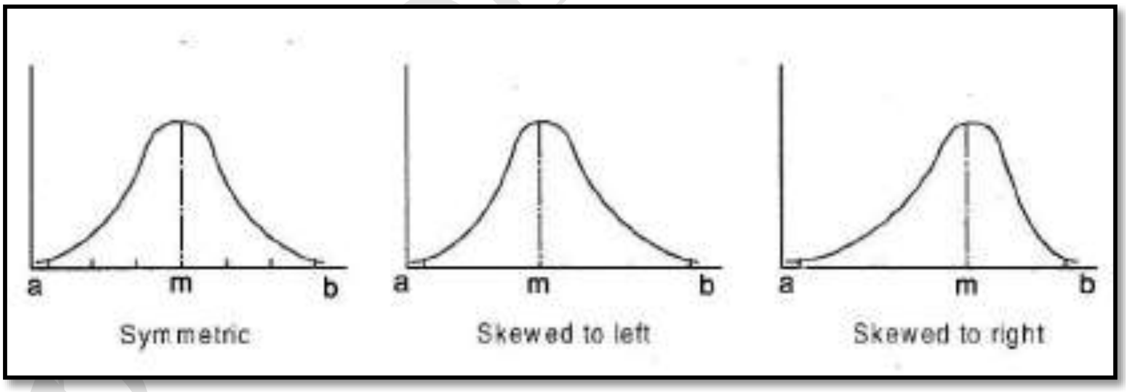
PERT differs from CPM in that it assumes **probabilistic duration times** based on three estimates while the time estimates for CPM are deterministic.

1- **Optimistic time, a,** and **the** estimate of the activity's duration under the most favorable conditions which is the shortest possible time required for the completion of an activity.

2- **Pessimistic time, b,** and the estimate of the activity's duration under the worse conditions which is the maximum possible time the activity will take if everything goes bad.

3- **Most likely time, m,** most likely value for the activity duration and estimate under normal conditions. Value of m falls in the range (a, b).

The best way of predicting the three-time estimates is by intelligent guessing. The experienced person who may be an engineer, foreman, or worker having sufficient technical competence is asked to guess the various time estimates. The probabilistic nature of the activity times is described by the beta distribution. The three estimates (a, b, m) will be determined by the mean and variance of the beta distribution.



Based on the estimates, the mean (expected time) **average duration time**, $\bar{D}$, and **variance**, v, are approximated as

$$\bar{D} = \frac{a + b + 4m}{6}$$

$$v = \sigma^2 = \left(\frac{b - a}{6}\right)^2$$

Notes for solve by PERT method

1. Prepare a list of all activities (Time, human resources, money, and material) involved in the project.
2. Draw a PERT diagram as per the rules discussed earlier for drawing a network diagram.
3. Number all events in ascending order from left to right.
4. The estimated average activity time is $\overline{D}_{ij}$ and the variance v_{ij}.
5. Find the Critical Path CP which is the total duration of the activities.
6. Find the variance v of each activity time of the CP.
7. Find the standard deviation of critical path $\{v\,(\mathrm{CP})\}$ is:

$$\sigma = \sqrt{v(CP)} = \sqrt{\sum_{i=1}^{n} v_i}$$

where v_i is a variance of critical activities.

Let CP be the random variable denoting that the total duration of the activities on a critical path. Applying the assumption that CP is normally distributed, we can answer questions such as the following: What is the probability that the project will be completed within a certain time

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Estimation of project completion time using standard normal variance value.

Calculate the probability of completing the project or before a specified completion date by using the standard normal equation shown below:

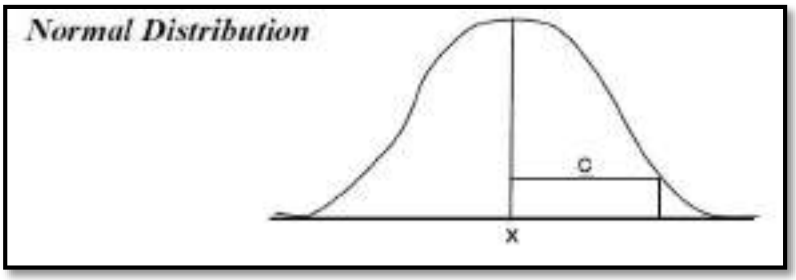
$$C = \left(\frac{K - X}{\sigma}\right)$$

Where

K = Proposed project completion time.

X = Expected time for the critical path.

σ = the standard deviation of critical path.



The probability that the project will be completed in CP duration time is

P(project finishes within K time) = **P(CP ≤ X) = P(Z ≤ C)**
Where Z is a random variable with standard normal distribution

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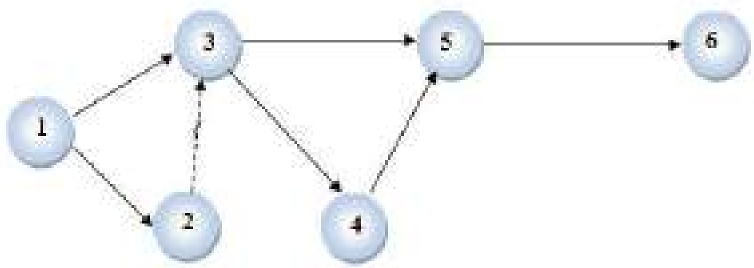
Example1: Consider the following table with **a, b, m** for activities of the network down. Find the probabilities that the different nodes of the project are realized without delay? What is the probability that the project will be completed within 35 days?



activities	a	b	m
1-2	**5**	**13**	**9**
1-3	**2**	**10**	**6**
3-5	**3**	**13**	**8**
3-4	**1**	**13**	**7**
4-5	**8**	**12**	**10**
5-6	**9**	**15**	**12**

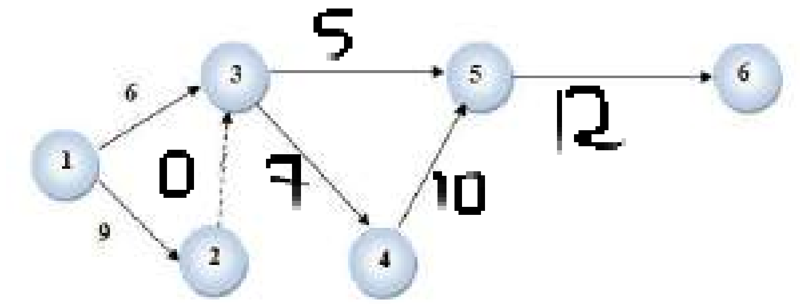
Solution:

1- Find average activities time and variance for all the activities.

activities	$\overline{D}_{ij}$	v_{ij}
1-2	**9**	**1.78**
1-3	**6**	**1.78**
3-5	**8**	**2.78**
3-4	**7**	**4**
4-5	**10**	**0.44**
5-6	**12**	**1**

Note that dummy activity with (a, b, m) = (0, 0, 0) has zero mean and variance. So $\overline{D}_{23} = v_{23} = 0$.

A network diagram with the average activities time is:



2- Determine the critical path for the project.

Critical Path: 1→2→3→4→5→6 ⇒ (has longest path)

3- Determine the expected completion time for the project.

$\overline{D}$ For CP = 9 + 0 + 7 + 10 + 12 = 38

The expected completion time for the project is =38 days

4- Find variance for each activity for the CP for the project.

$V = V(c.p) = \sum V_{ij}$, V_{ij} is a variance of critical activities

Variance for CP = 1.78 + 0 + 4 + 0.44 + 1 = 7.22

5- Find the Standard deviation of the critical path.

The standard deviation for CP is

$$\sigma = \sqrt{7.22} = 2.69.$$

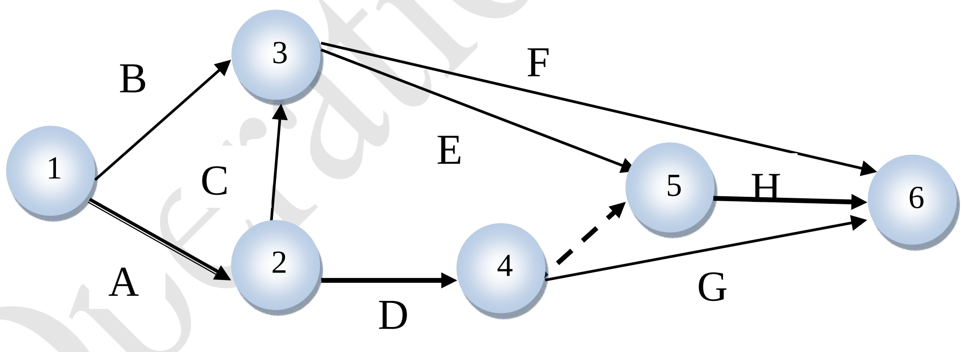
6- **The probability that the project will be completed within 35 is just P(CP ≤ 35) days.**

K = 35
X = 38
σ = Standard deviation of critical path = 2.69

$$C = \left(\frac{K - X}{\sigma}\right) = \left(\frac{35 - 38}{2.69}\right) = -1.12$$

P (CP ≤ 35) = P (Z ≤ -1.12) = 0.1335 (from normal distribution table)
This means that there is a 13 .35% chance that the project will be completed within 35 days.

Example2: Consider the following project with the estimates **a, b, m** for activities. Find the probabilities that the different nodes of the project are realized without delay. What is the probability that the project will be completed within 20 days? 30 days?



activities	a	b	m
1-2	**3**	**7**	**5**
1-3	**4**	**8**	**6**
2-3	**1**	**5**	**3**
2-4	**5**	**11**	**8**

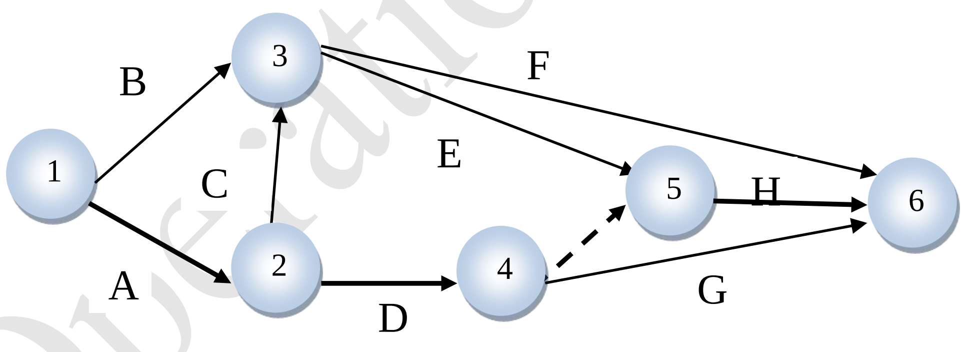
3-5	**1**	**3**	**2**
3-6	**9**	**13**	**11**
4-6	**1**	**1**	**1**
5-6	**10**	**14**	**12**

Solution:

1- Find average activities time and variance for all the activities.

activities	$\overline{D}_{ij}$	v_{ij}
1-2	**5**	**0.444**
1-3	**6**	**0.444**
2-3	**3**	**0.444**
2-4	**8**	**1.000**
3-5	**2**	**0.111**
3-6	**11**	**0.444**
4-6	**1**	**0.000**
5-6	**12**	**0.444**

Note that $\overline{D}_{45} = v_{45} = 0$. A network diagram with the average activities time is:



1- Determine the critical path for the project using the average activity time.

Critical Path: 1→2→4→5→6 ⇒ (has the longest path)

2- Determine the expected completion time for the project.

$\overline{D}$ For CP = 5 + 8 + 0 + 12 = 25

The expected completion time for the project is = 25 days

3- Find variance for each activity for the CP for the project.

$$V = V(c.p) = \sum V_{ij}$$, V_{ij} is a variance of critical activities

Variance for CP = 0.444 + 1 + 0 + 0.444 = 1.888

4- Find the Standard deviation of the critical path.

The standard deviation for CP is

$$\sigma = \sqrt{1.888} = 1.37$$

The probability that the project will be completed within 30 is just P(CP ≤ 30) days.

K = 20
X = 25
σ = Standard deviation of critical path = 1.37

$$C = \left(\frac{K - X}{\sigma}\right) = \left(\frac{20 - 25}{1.37}\right) = -3.6493$$

P (CP ≤ 20) = P (Z ≤ -3.649) = 0.0002 (from normal distribution table). This means that there is a 0.02 % chance that the project will be completed within 20 days.

The probability that the project will be completed within 30 is just P(CP ≤ 30) days.

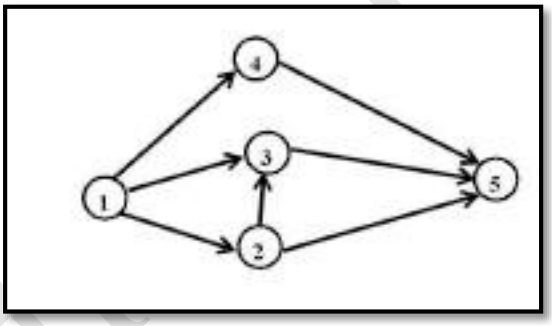
$$C = \left(\frac{K - X}{\sigma}\right) = \left(\frac{30 - 25}{1.37}\right) = 3.6493$$

P (CP ≤ 30) = P (Z ≤ 3.649) = 0.9998 (from normal distribution table)
This means that there is a 99.98% chance that the project will be completed within 30 days.

1- **H.W: C**onstruct a network diagram for a project consisting of the following estimates of activity times (weeks). What is the probability that the project will be completed within 18 weeks? , 11 weeks?

activities	pessimistic time b	optimistic time a	most probable time (m)
1-2	**6**	**2**	**4**
1-3	**7**	**6**	**5**
1-4	**8**	**6**	**4**
2-3	**6**	**1**	**5**
2-5	**3**	**1**	**2**
3-5	**7**	**4**	**7**
4-5	**3**	**1**	**2**

Solution: The network diagram is:



Game Theory

Game theory is useful for making decisions in cases where two or more decision-makers have conflicting interests. Our lecture deals with situations where there are only two decision-makers (or players). How to select the optimal strategy for each player without knowledge of the competitor's strategy is the basic problem of playing a game? The theory of games is based on the minimax principle which implies that each competitor will act to minimize has a maximum loss (or maximize his minimum gain) or achieve the best of the worst.

Some important definitions:

1- **Games:** A game is defined as an activity between two or more persons involving activities by each person according to a set of rules, at the end of which each person gets some benefit or suffers a loss.
2- **Play:** The set of rules defines the game; going through the set of rules once by participants defines a play.
3- **Chance of strategy:** If in a game, activities are determined by skill, it is said to be a game of strategy; if they are determined by chance, it is a **Game of Chance.**
4- **Two-person game:** Suppose there is a number (n) of players such that $n \geq 2$. In the case of $n = 2$, it is called a Two-person game.
5- **Strategy:** A strategy for a given player is a set of rules that specifies which of the available courses of action he should make at each play.

There are two types of strategy

i- **Pure strategy:** If players know exactly what the other player is going to do, a deterministic situation is obtained and the objective is to maximize the gain.
ii- **Mixed strategy:** If a player is guessing as to which activity is to be selected by the other player, a probabilistic situation is obtained and the objective function is to maximize the expected gain.

6- **Two people Zero-Sum Games (Rectangular Games):** A game with only two players (say **A** and **B**). If the losses of one player are equivalent to the gains of the other, so that the sum of their net gains is zero. The resulting gain is represented by pay-off-matrix in rectangular form.

7- **Pay-off-matrix (gain matrix):** it is a table that shows how payment should be made at the end of a play. The representation indicates that if **A** uses strategy i and **B** uses strategy j, the payoff to **A** is a_{ij}, which means that the payoff to **B** is $-a_{ij}$.

8- Saddle point: It is a position of such an element in the pay-off matrix, which is the **minimum in its row and maximum in its column**. The payoff at the saddle point is called the **value of the game**.

$$
\begin{array}{cc|cccc}
 & & & \text{player B} & & \\
 & & 1 & 2 & 3\ j \ldots & n \\
\hline
 & 1 & a_{11} & a_{12} & \cdots & a_{1n} \\
\text{player A} & 2 & a_{21} & a_{22} & \cdots & a_{2n} \\
 & 3 & & & & \\
 & i & \vdots & \vdots & \vdots & \vdots \\
 & \vdots & & & & \\
 & m & a_{m1} & a_{m2} & \cdots & a_{mn}
\end{array}
$$

9- Rule of Dominance:

a- If all elements of a row, say ith, are less than or equal to the corresponding elements of any other row, say jth row, then ith row is dominated by jth row.

b- If all elements of a column, say ith, are greater than or equal to the corresponding elements of any other column, say jth, then ith column is dominated by jth column.

c- A pure strategy may be dominated if it is inferior to an average of two or more other pure strategies.

Pure Strategies: Games with Saddle Point

Player **A** whose strategies are repressed by the rows. The criterion calls for **A** is to select the strategy that maximizes his minimum gains. So we call him maximized. Player **B** in turn tries to minimize his maximum losses and is called the minimizer. They do it as

1- Player **A** chooses minimum value in each row (least gain or payoff to player **A**), then he chooses maximin strategy to maximize his minimum gains.

2- Player **B** wants to minimize his losses, so he chooses maximum loss value in each column, and then he chooses a minimax strategy to minimize his maximum losses.

If the **maximin value equals the minimax value**, then the game is said to have a **saddle or equilibrium point** and the corresponding strategies are called optimal strategies.

Example1: Solve the game whose payoff matrix is given by.

$$
\begin{array}{cc|ccc|}
 & & & B & \\
 & & I & II & III \\
\hline
 & I & -2 & 15 & -2 \\
A & II & -5 & -6 & -4 \\
 & III & -5 & 20 & -8 \\
\hline
\end{array}
$$

Solution:

$$
\begin{array}{cc|ccc|c}
 & & & B & & \\
 & & I & II & III & \\
\hline
 & I & -2 & 15 & -2 & -2 \\
A & II & -5 & -6 & -4 & -6 \\
 & III & -5 & 20 & -8 & -8 \\
\hline
 & & -2 & 20 & -2 &
\end{array}
$$

1- The best strategy for **A** is I,
2- The best strategy for **B** is either I or II,
3- The value of the game is -2 for **A** and +2 for **B**.

Example2: Solve the following game whose payoff matrix is given by.

$$
\begin{array}{cc|ccc|}
 & & & B & \\
 & & I & II & III \\
\hline
A & I & 6 & 8 & 6 \\
 & II & 4 & 12 & 2 \\
\hline
\end{array}
$$

Solution:

1- The best strategy for **A** is ……**I**….

2- The best strategy for **B** is either I or II ……

3- The value of the game is ……6….. for **A** and ……-6… for **B**.

H.W: Solve the following game whose payoff matrix is given by.

$$
\begin{array}{cc|ccc|}
 & & & B & \\
 & & I & II & III \\
\hline
 & I & -3 & -2 & 6 \\
A & II & 2 & 0 & 2 \\
 & III & 5 & -2 & -4 \\
\hline
\end{array}
$$

GAMES WITH NO SADDLE POINT

LESSON OUTLINE

- The concept of a 2x2 game with no saddle point
- The method of solution

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the concept of a 2x2 game with no saddle point
- know the method of solution of a 2x2 game without saddle point
- solve a game with a given payoff matrix
- interpret the results obtained from the payoff matrix

2 x 2 zero-sum game

When each one of the first player A and the second player B has exactly two strategies, we have a 2 x 2 game.

Motivating point

First let us consider an illustrative example.

Problem 1:

Examine whether the following 2 x 2 game has a saddle point

$$
\begin{array}{cc}
 & \text{Player B} \\
\text{Player A} & \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}
\end{array}
$$

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	3
2	2

Maximum of {3, 2} = 3

Next consider the maximum of each column.

Column	Maximum Value
1	4
2	5

Minimum of {4, 5} = 4

We see that max {row minima} and min {column maxima} are not equal. Hence the game has no saddle point.

Method of solution of a 2x2 zero-sum game without saddle point

Suppose that a 2x2 game has no saddle point. Suppose the game has the following pay-off matrix.

$$
\begin{array}{cc}
 & \text{Player B} \\
 & \text{Strategy} \\
\text{Player A Strategy} & \boxed{\begin{array}{cc} a & b \\ c & d \end{array}}
\end{array}
$$

Since this game has no saddle point, the following condition shall hold:

$$\text{Max}\{\text{Min}\{a,b\},\text{Min}\{c,d\}\} \neq \text{Min}\{\text{Max}\{a,c\},\text{Max}\{b,d\}\}$$

In this case, the game is called a mixed game. No strategy of Player A can be called the best strategy for him. Therefore A has to use both of his strategies. Similarly no strategy of Player B can be called the best strategy for him and he has to use both of his strategies.

Let p be the probability that Player A will use his first strategy. Then the probability that Player A will use his second strategy is 1-p.

<u>If Player B follows his first strategy</u>

$$
\begin{aligned}
&\text{Expected value of the pay-off to Player A} \\
&= \left\{ \begin{array}{c} \text{Expected value of the pay-off to Player A} \\ \text{arising from his first strategy} \end{array} \right\} + \left\{ \begin{array}{c} \text{Expected value of the pay-off to Player A} \\ \text{arising from his second strategy} \end{array} \right\} \\
&= ap + c(1-p) \qquad (1)
\end{aligned}
$$

In the above equation, note that the expected value is got as the product of the corresponding values of the pay-off and the probability.

<u>If Player B follows his second strategy</u>

$$\left.\begin{array}{c} \text{Expected value of the} \\ \text{pay-off to Player A} \end{array}\right\} = bp + d(1-p) \qquad (2)$$

If the expected values in equations (1) and (2) are different, Player B will prefer the minimum of the two expected values that he has to give to player A. Thus B will have a pure strategy. This contradicts our assumption that the game is a mixed one. Therefore the expected values of the pay-offs to Player A in equations (1) and (2) should be equal. Thus we have the condition

$$\begin{aligned}
ap + c(1-p) &= bp + d(1-p)\\
ap - bp &= (1-p)[d-c]\\
p(a-b) &= (d-c) - p(d-c)\\
p(a-b) + p(d-c) &= d-c\\
p(a-b+d-c) &= d-c\\
p &= \frac{d-c}{(a+d)-(b+c)}\\
1-p &= \frac{a+d-b-c-d+c}{(a+d)-(b+c)}\\
&= \frac{a-b}{(a+d)-(b+c)}
\end{aligned}$$

$$\left\{\begin{matrix}\text{The number of times A}\\ \text{will use first strategy}\end{matrix}\right\} : \left\{\begin{matrix}\text{The number of times A}\\ \text{will use second strategy}\end{matrix}\right\} = \frac{d-c}{(a+d)-(b+c)} : \frac{a-b}{(a+d)-(b+c)}$$

The expected pay-off to Player A

$$\begin{aligned}
&= ap + c(1-p)\\
&= c + p(a-c)\\
&= c + \frac{(d-c)(a-c)}{(a+d)-(b+c)}\\
&= \frac{c\{(a+d)-(b+c)\} + (d-c)(a-c)}{(a+d)-(b+c)}\\
&= \frac{ac + cd - bc - c^2 + ad - cd - ac + c^2)}{(a+d)-(b+c)}\\
&= \frac{ad-bc}{(a+d)-(b+c)}
\end{aligned}$$

Therefore, the value V of the game is

$$\frac{ad-bc}{(a+d)-(b+c)}$$

To find the number of times that B will use his first strategy and second strategy:

Let the probability that B will use his first strategy be r. Then the probability that B will use his second strategy is 1-r.

<u>When A use his first strategy</u>

The expected value of loss to Player B with his first strategy = ar

The expected value of loss to Player B with his second strategy = b(1-r)

Therefore the expected value of loss to B = ar + b(1-r) (3)

<u>When A use his second strategy</u>

The expected value of loss to Player B with his first strategy = cr

The expected value of loss to Player B with his second strategy = d(1-r)

Therefore the expected value of loss to B = cr + d(1-r) (4)

If the two expected values are different then it results in a pure game, which is a contradiction. Therefore the expected values of loss to Player B in equations (3) and (4) should be equal. Hence we have the condition

$$
\begin{aligned}
ar + b(1-r) &= cr + d(1-r) \\
ar + b - br &= cr + d - dr \\
ar - br - cr + dr &= d - b \\
r(a - b - c + d) &= d - b \\
r &= \frac{d-b}{a-b-c+d} \\
&= \frac{d-b}{(a+d)-(b+c)}
\end{aligned}
$$

Problem 2:

Solve the following game

$$
\begin{array}{cc}
 & Y \\
X & \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}
\end{array}
$$

Solution:

First consider the row minima.

Row	Minimum Value
1	2
2	1

Maximum of {2, 1} = 2

Next consider the maximum of each column.

Column	Maximum Value
1	4
2	5

Minimum of {4, 5} = 4

We see that

Max { row minima} $\neq$ min { column maxima}

So the game has no saddle point. Therefore it is a mixed game.

We have a = 2, b = 5, c = 4 and d = 1.

Let p be the probability that player X will use his first strategy. We have

$$p = \frac{d-c}{(a+d)-(b+c)}$$
$$= \frac{1-4}{(2+1)-(5+4)}$$
$$= \frac{-3}{3-9}$$
$$= \frac{-3}{-6}$$
$$= \frac{1}{2}$$

The probability that player X will use his second strategy is 1-p = $1-\frac{1}{2} = \frac{1}{2}$.

Value of the game V = $\frac{ad-bc}{(a+d)-(b+c)} = \frac{2-20}{3-9} = \frac{-18}{-6} = 3$.

Let r be the probability that Player Y will use his first strategy. Then the probability that Y will use his second strategy is (1-r). We have

$$r = \frac{d-b}{(a+d)-(b+c)}$$
$$= \frac{1-5}{(2+1)-(5+4)}$$
$$= \frac{-4}{3-9}$$
$$= \frac{-4}{-6}$$
$$= \frac{2}{3}$$
$$1-r = 1-\frac{2}{3} = \frac{1}{3}$$

Interpretation.

p : (1-p) = $\frac{1}{2} : \frac{1}{2}$

Therefore, out of 2 trials, player X will use his first strategy once and his second strategy once.

$r : (1-r) = \frac{2}{3} : \frac{1}{3}$

Therefore, out of 3 trials, player Y will use his first strategy twice and his second strategy once.

QUESTIONS

1. What is a 2x2 game with no saddle point? Explain.
2. Explain the method of solution of a 2x2 game without saddle point.
3. Solve the following game

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 12 & 4 \\ 3 & 7 \end{bmatrix} \end{array}$$

Answer: $p = \frac{1}{3}$, $r = \frac{1}{4}$, $V = 6$

THE PRINCIPLE OF DOMINANCE

LESSON OUTLINE

- The principle of dominance
- Dividing a game into sub games

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the principle of dominance
- solve a game using the principle of dominance
- solve a game by dividing a game into sub games

The principle of dominance

In the previous lesson, we have discussed the method of solution of a game without a saddle point. While solving a game without a saddle point, one comes across the phenomenon of the dominance of a row over another row or a column over another column in the pay-off matrix of the game. Such a situation is discussed in the sequel.

In a given pay-off matrix A, we say that the i^{th} row dominates the k^{th} row if

$$a_{ij} \geq a_{kj} \quad \text{for all } j = 1, 2, \ldots, n$$

and

$$a_{ij} > a_{kj} \quad \text{for at least one } j.$$

In such a situation player A will never use the strategy corresponding to k^{th} row, because he will gain less for choosing such a strategy.

Similarly, we say the p^{th} column in the matrix dominates the q^{th} column if

$a_{ip} \leq a_{iq}$ for all $i = 1, 2, \ldots, m$

and

$a_{ip} < a_{iq}$ for at least one i.

In this case, the player B will loose more by choosing the strategy for the q^{th} column than by choosing the strategy for the p^{th} column. So he will never use the strategy corresponding to the q^{th} column. When dominance of a row (or a column) in the pay-off matrix occurs, we can delete a row (or a column) from that matrix and arrive at a reduced matrix. This principle of dominance can be used in the determination of the solution for a given game.

Let us consider an illustrative example involving the phenomenon of dominance in a game.

Problem 1:

Solve the game with the following pay-off matrix:

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \text{Player A} \quad \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 4 & 2 & 3 & 6 \\ 3 & 4 & 7 & 5 \\ 6 & 3 & 5 & 4 \end{bmatrix} \end{array}$$

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	2
2	3
3	3

Maximum of {2, 3, 3} = 3

Next consider the maximum of each column.

Column	Maximum Value
1	6
2	4
3	7
4	6

Minimum of {6, 4, 7, 6} = 4

The following condition holds:

$$\text{Max \{row minima\}} \neq \text{min \{column maxima\}}$$

Therefore we see that there is no saddle point for the game under consideration.

Compare columns II and III.

Column II	Column III
2	3
4	7
3	5

We see that each element in column III is greater than the corresponding element in column II. The choice is for player B. Since column II dominates column III, player B will discard his strategy 3.

Now we have the reduced game

$$\begin{array}{c c} & \begin{array}{ccc} \mathrm{I} & \mathrm{II} & \mathrm{IV} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 4 & 2 & 6 \\ 3 & 4 & 5 \\ 6 & 3 & 4 \end{bmatrix} \end{array}$$

For this matrix again, there is no saddle point. Column II dominates column IV. The choice is for player B. So player B will give up his strategy 4

The game reduces to the following:

$$\begin{array}{c c} & \begin{array}{cc} \mathrm{I} & \mathrm{II} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 4 & 2 \\ 3 & 4 \\ 6 & 3 \end{bmatrix} \end{array}$$

This matrix has no saddle point.

The third row dominates the first row. The choice is for player A. He will give up his strategy 1 and retain strategy 3. The game reduces to the following:

$$\begin{bmatrix} 3 & 4 \\ 6 & 3 \end{bmatrix}$$

Again, there is no saddle point. We have a 2x2 matrix. Take this matrix as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then we have a = 3, b = 4, c = 6 and d = 3. Use the formulae for p, 1-p, r, 1-r and V.

$$p = \frac{d-c}{(a+d)-(b+c)}$$

$$= \frac{3-6}{(3+3)-(6+4)}$$

$$= \frac{-3}{6-10}$$

$$= \frac{-3}{-4}$$

$$= \frac{3}{4}$$

$$1-p = 1-\frac{3}{4} = \frac{1}{4}$$

$$r = \frac{d-b}{(a+d)-(b+c)}$$
$$= \frac{3-4}{(3+3)-(6+4)}$$
$$= \frac{-1}{6-10}$$
$$= \frac{-1}{-4}$$
$$= \frac{1}{4}$$
$$1-r = 1-\frac{1}{4} = \frac{3}{4}$$

The value of the game

$$V = \frac{ad-bc}{(a+d)-(b+c)}$$
$$= \frac{3x3-4x6}{-4}$$
$$= \frac{-15}{-4}$$
$$= \frac{15}{4}$$

Thus, $X = \left(\frac{3}{4}, \frac{1}{4}, 0, 0\right)$ and $Y = \left(\frac{1}{4}, \frac{3}{4}, 0, 0\right)$ are the optimal strategies.

Method of convex linear combination

A strategy, say s, can also be dominated if it is inferior to a convex linear combination of several other pure strategies. In this case if the domination is strict, then the strategy s can be deleted. If strategy s dominates the convex linear combination of some other pure strategies, then one of the pure strategies involved in the combination may be deleted. The domination will be decided as per the above rules. Let us consider an example to illustrate this case.

Problem 2:

Solve the game with the following pay-off matrix for firm A:

$$\text{Firm A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \text{Firm B} \\ \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 & 6 \\ 4 & 0 & 6 & 8 & 5 \\ -2 & -6 & -4 & 4 & 2 \\ 4 & -3 & 5 & 6 & 3 \\ 4 & -1 & 5 & 7 & 3 \end{bmatrix} \end{array}$$

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	-2
2	0
3	-6
4	-3
5	-1

Maximum of {-2, 0, -6, -3, -1} = 0

Next consider the maximum of each column.

Column	Maximum Value
1	4
2	8
3	6
4	8
5	6

Minimum of { 4, 8, 6, 8, 6} = 4

Hence,

Maximum of {row minima} $\neq$ minimum of {column maxima}.

So we see that there is no saddle point. Compare the second row with the fifth row. Each element in the second row exceeds the corresponding element in the fifth row. Therefore, A_2 dominates A_5. The choice is for firm A. It will retain strategy A_2 and give up strategy A_5. Therefore the game reduces to the following.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 & 6 \\ 4 & 0 & 6 & 8 & 5 \\ -2 & -6 & -4 & 4 & 2 \\ 4 & -3 & 5 & 6 & 3 \end{bmatrix} \end{array}$$

Compare the second and fourth rows. We see that A_2 dominates A_4. So, firm A will retain the strategy A_2 and give up the strategy A_4. Thus the game reduces to the following:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 & 6 \\ 4 & 0 & 6 & 8 & 5 \\ -2 & -6 & -4 & 4 & 2 \end{bmatrix} \end{array}$$

Compare the first and fifth columns. It is observed that B_1 dominates B_5. The choice is for firm B. It will retain the strategy B_1 and give up the strategy B_5. Thus the game reduces to the following

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 \\ 4 & 0 & 6 & 8 \\ -2 & -6 & -4 & 4 \end{bmatrix} \end{array}$$

Compare the first and fourth columns. We notice that B_1 dominates B_4. So firm B will discard the strategy B_4 and retain the strategy B_1. Thus the game reduces to the following:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 \\ 4 & 0 & 6 \\ -2 & -6 & -4 \end{bmatrix} \end{array}$$

For this reduced game, we check that there is no saddle point.

Now none of the pure strategies of firms A and B is inferior to any of their other strategies. But, we observe that convex linear combination of the strategies B_2 and B_3 dominates B_1, i.e. the averages of payoffs due to strategies B_2 and B_3,

$$\left\{\frac{8-2}{2}, \frac{0+6}{2}, \frac{-6-4}{2}\right\} = \{3, 3, -5\}$$

dominate B_1. Thus B_1 may be omitted from consideration. So we have the reduced matrix

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{bmatrix} 8 & -2 \\ 0 & 6 \\ -6 & -4 \end{bmatrix} \end{array}$$

Here, the average of the pay-offs due to strategies A_1 and A_2 of firm A, i.e. $\left\{\frac{8+0}{2}, \frac{-2+6}{2}\right\} = \{4, 2\}$ dominates the pay-off due to A_3. So we get a new reduced 2x2 pay-off matrix.

$$\begin{array}{c|c} & \text{Firm B's strategy} \\ & \begin{array}{cc} B_2 & B_3 \end{array} \\ \text{Firm A's strategy} & \begin{array}{c} A_1 \\ A_2 \end{array} \begin{bmatrix} 8 & -2 \\ 0 & 6 \end{bmatrix} \end{array}$$

We have a = 8, b = -2, c = 0 and d = 6.

$$p = \frac{d-c}{(a+d)-(b+c)}$$
$$= \frac{6-0}{(6+8)-(-2+0)}$$
$$= \frac{6}{16}$$
$$= \frac{3}{8}$$
$$1-p = 1-\frac{3}{8} = \frac{5}{8}$$

$$r = \frac{d-b}{(a+d)-(b+c)}$$
$$= \frac{6-(-2)}{16}$$
$$= \frac{8}{16}$$
$$= \frac{1}{2}$$
$$1-r = 1-\frac{1}{2} = \frac{1}{2}$$

Value of the game:

$$V = \frac{ad-bc}{(a+d)-(b+c)}$$
$$= \frac{6x8-0x(-2)}{16}$$
$$= \frac{48}{16} = 3$$

So the optimal strategies are

$$A = \left\{\frac{3}{8}, \frac{5}{8}, 0, 0, 0\right\} \text{ and } B = \left\{0, \frac{1}{2}, \frac{1}{2}, 0, 0\right\}.$$

The value of the game = 3. Thus the game is favourable to firm A.

Problem 3:

For the game with the following pay-off matrix, determine the saddle point

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \text{Player A} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 2 & -1 & 0 & -3 \\ 1 & 0 & 3 & 2 \\ -3 & -2 & -1 & 4 \end{bmatrix} \end{array}$$

Solution:

$$\begin{array}{cccc} & \text{Column II} & \text{Column III} & \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{array}{|c|} \hline -1 \\ 0 \\ -2 \\ \hline \end{array} & \begin{array}{|c|} \hline 0 \\ 3 \\ -1 \\ \hline \end{array} & \begin{array}{c} 0 > -1 \\ 3 > 0 \\ -1 > -2 \end{array} \end{array}$$

The choice is with the player B. He has to choose between strategies II and III. He will lose more in strategy III than in strategy II, irrespective of what strategy is followed by A. So he will drop strategy III and retain strategy II. Now the given game reduces to the following game.

$$\begin{array}{cc} & \begin{array}{ccc} \text{I} & \text{II} & \text{IV} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & -1 & -3 \\ 1 & 0 & 2 \\ -3 & -2 & 4 \end{bmatrix} \end{array}$$

Consider the rows and columns of this matrix.

Row minimum:

I Row	:	-3	
II Row	:	0	Maximum of {-3, 0, -3} = 0
III Row	:	-3	

Column maximum:

I Column	:	2	
II Column	:	0	Minimum of {2, 0, 4} = 0
III Column	:	4	

We see that

Maximum of row minimum = Minimum of column maximum = 0.

So, a saddle point exists for the given game and the value of the game is 0.

Interpretation:

No player gains and no player loses. i.e., The game is not favourable to any player. i.e. It is a fair game.

Problem 4:

Solve the game

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 4 & 8 & 6 \\ 6 & 2 & 10 \\ 4 & 5 & 7 \end{bmatrix} \end{array}$$

Solution:

First consider the minimum of each row.

Row	Minimum
1	4
2	2
3	4

Maximum of $\{4, 2, 4\} = 4$

Next, consider the maximum of each column.

Column	Maximum
1	6
2	8
3	10

Minimum of $\{6, 8, 10\} = 6$

Since Maximum of { Row Minima} and Minimum of { Column Maxima } are different, it follows that the given game has no saddle point.

Denote the strategies of player A by A_1, A_2, A_3. Denote the strategies of player B by B_1, B_2, B_3.

Compare the first and third columns of the given matrix.

$$\begin{array}{cc} B_1 & B_3 \\ \boxed{\begin{matrix} 4 \\ 6 \\ 7 \end{matrix}} & \boxed{\begin{matrix} 6 \\ 10 \\ 7 \end{matrix}} \end{array}$$

The pay-offs in B_3 are greater than or equal to the corresponding pay-offs in B_1. The player B has to make a choice between his strategies 1 and 3. He will lose more if he follows strategy 3 rather than strategy 1. Therefore he will give up strategy 3 and retain strategy 1. Consequently, the given game is transformed into the following game:

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 4 & 5 \end{bmatrix} \end{array}$$

Compare the first and third rows of the above matrix.

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_3 \end{array} & \begin{bmatrix} 4 & 8 \\ 4 & 5 \end{bmatrix} \end{array}$$

The pay-offs in A_1 are greater than or equal to the corresponding pay-offs in A_3. The player A has to make a choice between his strategies 1 and 3. He will gain more if he follows strategy 1 rather than strategy 3. Therefore he will retain strategy 1 and give up strategy 3. Now the given game is transformed into the following game.

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix} \end{array}$$

It is a 2x2 game. Consider the row minima.

Row	Minimum
1	4
2	2

Maximum of $\{4, 2\} = 4$

Next, consider the maximum of each column.

Column	Maximum
1	6
2	8

Minimum of $\{6, 8\} = 6$

Maximum { row minima} and Minimum { column maxima } are not equal

Therefore, the reduced game has no saddle point. So, it is a mixed game

Take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix}$. We have a = 4, b = 8, c = 6 and d = 2.

The probability that player A will use his first strategy is p. This is calculated as

$$p = \frac{d-c}{(a+d)-(b+c)}$$
$$= \frac{2-6}{(4+2)-(8+6)}$$
$$= \frac{-4}{6-14}$$
$$= \frac{-4}{-8} = \frac{1}{2}$$

The probability that player B will use his first strategy is r. This is calculated as

$$r = \frac{d-b}{(a+d)-(b+c)}$$
$$= \frac{2-8}{-8}$$
$$= \frac{-6}{-8}$$
$$= \frac{3}{4}$$

Value of the game is V. This is calculated as

$$V = \frac{ad-bc}{(a+d)-(b+c)}$$
$$= \frac{4x2-8x6}{-8}$$
$$= \frac{8-48}{-8}$$
$$= \frac{-40}{-8} = 5$$

Interpretation

Out of 3 trials, player A will use strategy 1 once and strategy 2 once. Out of 4 trials, player B will use strategy 1 thrice and strategy 2 once. The game is favourable to player A.

Problem 5: Dividing a game into sub-games

Solve the game with the following pay-off matrix.

$$\text{Player A} \quad \begin{array}{c c} & \text{Player B} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} & \begin{bmatrix} -4 & 6 & 3 \\ -3 & 3 & 4 \\ 2 & -3 & 4 \end{bmatrix} \end{array}$$

Solution:

First, consider the row mimima.

Row	Minimum
1	-4
2	-3
3	-3

Maximum of {-4, -3, -3} = -3

Next, consider the column maxima.

Column	Maximum
1	2
2	6
3	4

Minimum of {2, 6, 4} = 2

We see that Maximum of { row minima} $\neq$ Minimum of { column maxima}.

So the game has no saddle point. Hence it is a mixed game. Compare the first and third columns.

$$\begin{array}{ccc} \text{I Column} & \text{III Column} & \\ \begin{vmatrix} -4 \\ -3 \\ 2 \end{vmatrix} & \begin{vmatrix} 3 \\ 4 \\ 4 \end{vmatrix} & \begin{array}{r} -4 \le 3 \\ -3 \le 4 \\ 2 \le 4 \end{array} \end{array}$$

We assert that Player B will retain the first strategy and give up the third strategy. We get the following reduced matrix.

$$\begin{bmatrix} -4 & 6 \\ -3 & 3 \\ 2 & -3 \end{bmatrix}$$

We check that it is a game with no saddle point.

Sub games

Let us consider the 2x2 sub games. They are:

$$\begin{bmatrix} -4 & 6 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$$

First, take the sub game

$$\begin{bmatrix} -4 & 6 \\ -3 & 3 \end{bmatrix}$$

Compare the first and second columns. We see that $-4 \le 6$ and $-3 \le 3$. Therefore, the game reduces to $\begin{bmatrix} -4 \\ -3 \end{bmatrix}$. Since $-4 < -3$, it further reduces to ¯3.

Next, consider the sub game

$$\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$$

We see that it is a game with no saddle point. Take a = -4, b = 6, c = 2, d = -3. Then the value of the game is

$$\begin{aligned} V &= \frac{ad - bc}{(a+d)-(b+c)} \\ &= \frac{(-4)(-3)-(6)(2)}{(-4+3)-(6+2)} \\ &= 0 \end{aligned}$$

Next, take the sub game $\begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$. In this case we have a = -3, b = 3, c = 2 and d = -3. The value of the game is obtained as

$$\begin{aligned} V &= \frac{ad - bc}{(a+d)-(b+c)} \\ &= \frac{(-3)(-3)-(3)(2)}{(-3-3)-(3+2)} \\ &= \frac{9-6}{-6-5} = -\frac{3}{11} \end{aligned}$$

Let us tabulate the results as follows:

Sub game	Value
$\begin{bmatrix} -4 & 6 \\ -3 & 3 \end{bmatrix}$	-3
$\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$	0
$\begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$	$-\frac{3}{11}$

The value of 0 will be preferred by the player A. For this value, the first and third strategies of A correspond while the first and second strategies of the player B correspond to the value 0 of the game. So it is a fair game.

Graphical Method

When a $m \times n$ pay of matrix can be reduced to $m \times 2$ or $n \times 2$ pay off matrix, we can apply the sub game method. But too many sub games will be there it is time consuming. Hence, it is better to go for **Graphical method** to solve the game when we have $m \times 2$ or $n \times 2$ matrixes.

Problem 1

Solve the game whose pay of matrix is:

		B			
		I	II	III	IV
A	I	1	4	–2	–3
	II	2	1	4	5

Solution

Given payoff matrix is:

Solve the game whose pay of matrix is:

			B			
		I	II	III	IV	Row minimum
x	I	1	4	–2	–3	–3
A						
(1-x)	II	2	1	4	5	1
Column maximum		2	4	4	5	

No saddle point. If sub game method is to be followed, there will be many sub games. Hence, graphical method is used.

Let A play his first strategy with a probability of x, and then he has to play his second strategy with a probability of $(1 – x)$. Let us find the payoffs of A when B plays his various strategies.

Step 1

Find the payoffs of A when B plays his various strategies and A plays his first strategy with a probability x and second strategy with a probability $(1 – x)$. Let pay off be represented by P. Then A's payoffs, when

B plays his first strategy: $P_1 = 1\ x\ (x) + 2\ (1 – x) = 1x + 2 – 2x = 2 – x$.

B plays his second strategy: $P_2 = 4x + 1\ (1 – x) = 4x + 1 – x = 1 + 3x$.

B plays his third strategy: $P_3 = –2x + 4\ (1 – x) = – 2x + 4 –4x = 4 – 6x$.

B Plays his fourth strategy: $P_4 = -3x + 5\ (1 – x) = –3x + 5 – 5x = 5 – 8x$.

Step 2

All the above payoff equations are in the form of $y = mx + c$. Hence we can draw straight lines by giving various values to x. To do this let us write two **vertical lines,** keeping the distance between lines at least four centimeters. Then write a horizontal line to represent the probabilities. Let the left side vertical line represents, A's first strategy and the probability of $x = 1$ and right side vertical line represents A's second strategy and the probability of $1 – x$. Mark points 1, 2, 3 etc on vertical lines above the horizontal line and –1, –2, –3 etc, below the horizontal lines, to show the payoffs.

Step 3

By substituting $x = 0$ and $x = 1$ in payoff equations, mark the points on the lines drawn in step 2 above and joining the points to get the payoff lines.

Step 4

These lines intersect and form open polygon. These are known as upper bound above the horizontal line drawn and the open polygon below horizontal line is known as lower bound. The upper bound (open polygon above the horizontal line is used to find the decision of player B and the open polygon below the line is used to find the decision of player A. This we can illustrate by solving the numerical example given above.

Step 5

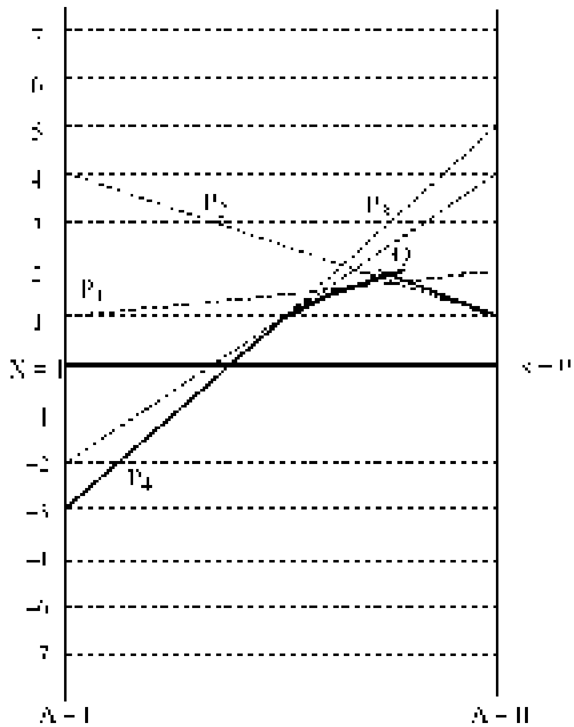
Remember that the objective of graphical method is also to reduce the given matrix to 2×2 matrix, so that we can apply the formula directly to get the optimal strategies of the players.

For $P_1 = 2 – x$, when $\times = 0$, $P_1 = 2$ and when $\times = 1$, $P_1 = 1$> Mark these points on the graph and join the points to get the line P_1. Similarly, we can write other profit lines.

$P_2 = 1 + 3x$, when $x = 0$, $P_2 = 1$, $x = 1$, $P_2 = 4$.

$P_3 = 4 – 6x$. When $x = 0$, $P_3 = 4$ and When $x = 1$, $P_3 = – 2$.

$P_4 = 5 – 8x$, When $x = 0$, $P_4 = 5$ and When $x = 1$, $P_4 = –3$.



After drawing the graph, the lower bound is marked, and the highest point of the lower bound is point Q, lies on the lines P_1 and P_2. Hence B plays the strategies II, and I so that he can minimize his losses. Now the game is reduced to 2 × 2 matrix. For this payoff matrix, we have to find optimal strategies of A and B. The reduced game is:

		B: I	B: II	Row minimum
A	I	1	4	1
A	II	2	1	1
Column Maximum:		2	4	

No saddle point. Hence we have to apply formula to get optimal strategies.

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - x_2 =}$

$x_1 = (1 - 2) / (1 + 1) - (4 + 2) = -1 / (2 - 6) = (-1 / -4) = 1 / 4$. and $x_2 = 1 - (1 / 4) = 3 / 4$

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - y_2}$

$y_1 = (1 - 4) / (-4) = (3 / 4)$, $y_2 = 1 - (3 / 4) = (1 / 4)$

Value of the game = $\boldsymbol{v = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) =}$
$(1 \times 1) - (4 \times 2) / -4 = (1 - 8) / -4 = -4 / -4 = (7 / 4)$
Answer: ***A* (1 / 4, 3 / 4), B (3 / 4, 1 / 4, 0, 0) , *v* = 7 /4. *A* always wins 7/4 units of money.**

Problem .24

Solve the given payoff matrix by Graphical method and state optimal strategies of players *A* and *B*.

		B 1	2	3	4	5
A	1	–5	5	0	–1	8
	2	8	–4	–1	6	–5

Solution

Given Payoff Matrix is

			B 1	2	3	4	5	Row minimum
x	A	1	–5	5	0	–1	8	–5
(1 – *x*)		2	8	–4	–1	6	–5	–5
Column Max:			8	5	0	6	8	

No saddle point. Reduce the given matrix by using graphical method. Let us write the payoff equations of *B* when he plays different strategies. *A* has only two strategies to use. Let us assume that *A* plays his first strategy with a probability *x* and his second strategy with a probability $(1 - x)$. The *B*'s payoffs are:

P_1 for *B*'s first strategy $= -5x + 8(1 - x)$, *i.e.* $P_1 = -5x + 8 - 8x = 8 - 13x$. When, $x = 0$, $P_1 = 8$, $x = 1$,

$P_1 = -5$.

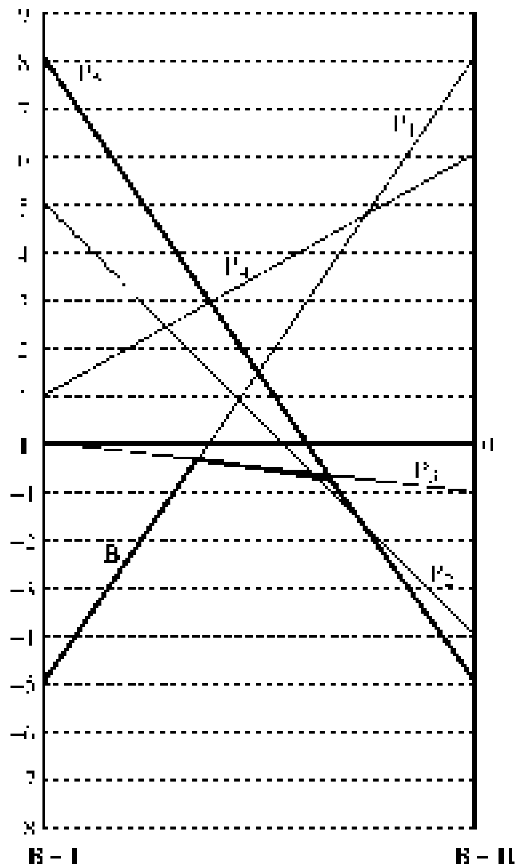
P_2 for *B*'s second Strategy $= 5x - 4(1 - x)$, *i.e.* $P_2 = 5x - 4 + 4x = 9x - 4$, When $x = 0$, $P_2 = -4$. When $x = 1$, $P_2 = 5$.

P_3 for *B*'s third strategy $= 0x - 1(1 - x) = x - 1$, When $x = 0$, $P_3 = -1$, and When $x = 1$, $P_3 = 0$.

P_4 for *B*'s fourth strategy $= -1x + 6(1 - x) = x + 6 - 6x = 6 - 5x$. When $x = 0$, $P_4 = 6$ and when $x = 1$, $P_4 = 1$

P_5 for *B*'s fifth strategy $= 8x - 5(1 - x) = 8x - 5 + 5x = 13x - 5$. When $x = 0$, $P_5 = -5$, when $x = 1$, $P_5 = 8$.

If we plot the above payoffs on the graph:



Now, player *B* has to select the strategies, as player *A* has only two strategies. To make A to get his minimum gains, *B* has to select the point *B* in the lower bound, which lies on both the strategies *B* –1 and *B* – 3. Hence now the 2 × 2 game is:

		B		
		1	3	Row minimum
A	1	–5	0	–5
	2	8	–1	–1
Column maximum.		8	0	

No saddle point. Hence apply the formula to get the optimal strategies.

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - x_2 =}$

$x_1 = [-1 - 8]/(-5 - 1) - (0 + 8) = (-9)/(-6 - 8) = (-9/-14) = (9/14)$ and $x_2 = [1 - (9/14) = (5/14)$

$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ or $= 1 - y_2$

$y_1 = [-1 - 0] / -(14) = -(-1 / -14) = (1 / 14)$ and $y_2 = 1 - (1 / 14) = (13 / 14)$

Value of the game $= v = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) = [(-1 \times -5) - (0 \times 8)] / (-14) = (-5 / -14) = (5 / 14)$.

Answer: $v = (5 / 14)$, A (9 / 14, 5 / 14), B (1/14, 0, 13 / 14, 0, 0). A always wins a sum of 5/14.

> **Note: While calculating the profits to draw graph, it is shown that first to write the equation and then substituting the values of 0 and 1 to x we can get the profits for each strategy. Students as well can directly write the profit points, without writing the equation. For example, in the given problem, we know that A plays his first strategy with x and then the second strategy with $(1- x)$ probability. When $x = 0$, the value is 8, *i.e.* the element a_{21} in the matrix. Similarly, when $x = 1$, the values is -5 *i.e.* the element a_{11}. We can write other values similarly. But it is advised it is not a healthy practice to write the values directly. At least show one equation and calculate the values and then write the other values directly. This is only a measure for emergency and not for regular practice.**

Problem 5

Solve the game graphically, whose pay off matrix is:

		B: I	B: II
	1	–6	7
	2	4	–5
A	3	–1	–2
	4	–2	5
	5	7	–6

Solution

The given pay off matrix is:

		B: I	B: II	Row minimum
	1	–6	7	–6
	2	4	–5	–5
A	3	–1	–2	–2
	4	–2	5	–2
	5	7	–6	–6
Column maximum:		7	7	

LESSON 8

LINEAR PROGRAMMING APPROACH TO GAME THEORY

LESSON OUTLINE

4. How to solve a game with LPP?
5. Formulation of LPP
6. Solution by simplex method

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the transformation of a game into LPP
- solve a game by simplex method

Introduction

When there is neither saddle point nor dominance in a problem of game theory and the payoff matrix is of order 3x3 or higher, the probability and graphical methods cannot be employed. In such a case, linear programming approach may be followed to solve the game.

Linear programming technique:

A general approach to solve a game by linear programming technique is presented below. Consider the following $m \times n$ game:

Player B

$$\text{Player A} \quad \begin{array}{c} \\ X_1 \\ X_2 \\ \\ X_m \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y_2 & Y_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ & & \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix} \end{array}$$

It is required to determine the optimal strategy for A = $\{X_1, X_2, ...X_m\}$ and B = $\{Y_1, Y_2, ...Y_n\}$. First we shall determine the optimal strategies of player B.

If player A adopts strategy X_1, then the expected value of loss to B is

$a_{11}Y_1 + a_{12}Y_2 + ... + a_{1n}Y_n \le V$,

where V is the value of game. If A adopts strategy X_2, then the expected value of loss to B is

$$a_{21}Y_1 + a_{22}Y_2 + ... + a_{2n}Y_n \leq V$$

and so on. Also we have

$$Y_1 + Y_2 + ... + Y_n = 1$$

and

$$Y_j \geq 0 \text{ for all j.}$$

Without loss of generality, we can assume that V > 0. Divide each of the above relation by V and let $Y'_j = \frac{Y_j}{V}$.

Then we have

$$\sum Y'_j = \sum \frac{Y_j}{V} = \frac{1}{V}$$

From this we obtain

$$a_{11}Y'_1 + a_{12}Y'_2 + ... + a_{1n}Y'_n \leq 1,$$
$$a_{21}Y'_1 + a_{22}Y'_2 + ... + a_{2n}Y'_n \leq 1,$$

$$a_{m1}Y'_1 + a_{m2}Y'_2 + ... + a_{mn}Y'_n \leq 1$$

and

$$Y'_1 + Y'_2 + ... + Y'_n = \frac{1}{V}$$

with $Y'_j \geq 0$ for all j.

The objective of player B is to minimise the loss to himself . Thus the problem is to minimize V, or equivalently to maximise $\frac{1}{V}$. Therefore, the objective of player B is to maximise the value of $Y'_1 + Y'_2 + ... + Y'_n$ subject to the m linear constraints provided above.

Statement of the problem:

Maximise: $Y'_1 + Y'_2 + ... + Y'_n$, subject to

$$a_{11}Y'_1 + a_{12}Y'_2 + ... + a_{1n}Y'_n \leq 1,$$
$$a_{21}Y'_1 + a_{22}Y'_2 + ... + a_{2n}Y'_n \leq 1,$$

$$a_{m1}Y'_1 + a_{m2}Y'_2 + ... + a_{mn}Y'_n \leq 1$$

$$Y'_1, Y'_2, ..., Y'_n \geq 0.$$

We can use simplex method to solve the above problem. For this purpose, we have to introduce non-negative slack variables $s_1, s_2, ..., s_m$ to each of the inequalities. So the problem can be restated as follows:

Restatement of the problem:

Maximise: $Y'_1 + Y'_2 + ... + Y'_n + 0s_1 + 0s_2 + ... + 0s_m$ subject to

$$a_{11}Y'_1 + a_{12}Y'_2 + ... + a_{1n}Y'_n + s_1 = 1,$$
$$a_{21}Y'_1 + a_{22}Y'_2 + ... + a_{2n}Y'_n + s_2 = 1,$$

$$a_{m1}Y'_1 + a_{m2}Y'_2 + ... + a_{mn}Y'_n + s_m = 1$$

with $Y_j = Y'_j V$ for all j and $s_1 \geq 0,\ s_2 \geq 0, ..., s_m \geq 0$.

Thus we get the optimal strategy for player B to be ($Y_1, Y_2, ..., Y_n$).

In a similar manner we can determine the optimal strategy for player A.

Application:

We illustrate the method for a 2X2 zero sum game.

Problem 1:

Solve the following game by simplex method for LPP:

Player B

$$\text{Player A} \quad \begin{bmatrix} 3 & 6 \\ 5 & 2 \end{bmatrix}$$

Solution :

Row minima : I row : 3

II row : 2

Maximum of {3,2} = 3

Column maxima: I column : 5

II column : 6

Minimum of {5,6} = 5

So, Maximum of {Row minima} $\neq$ Minimum of {Column maxima}.

Therefore the given game has no saddle point. It is a mixed game. Let us convert the given game into a LPP.

Problem formulation:

Let V denote the value of the game. Let the probability that the player B will use his first strategy be r and second strategy be s. Let V denote the value of the game.

When A follows his first strategy:

The expected payoff to A (i.e., the expected loss to B) = 3 r + 6 s.

This pay-off cannot exceed V. So we have

$$3r + 6s \le V \tag{1}$$

When A follows his second strategy:

The expected pay-off to A (i.e., expected loss to B) = 5 r + 2 s.

This cannot exceed V. Hence we obtain the condition

$$5r + 2s \le V \tag{2}$$

From (1) and (2) we have

$$3\frac{r}{V} + 6\frac{s}{V} \le 1$$

$$and\ \ 5\frac{r}{V} + 2\frac{s}{V} \le 1$$

Substitute $\frac{r}{V} = x,\ \frac{s}{V} = y$.

Then we have

$$3x + 6y \le 1$$

$$and\ \ 5x + 2y \le 1$$

where r and s are connected by the relation

$$r + s = 1.$$

$$\text{i.e., } \frac{r}{V} + \frac{s}{V} = \frac{1}{V}$$

$$\text{i.e., } x + y = \frac{1}{V}$$

B will try to minimise V. i.e., He will try to maximise $\frac{1}{V}$. Thus we have the following LPP.

Maximize $\frac{1}{V} = x + y$,

subject to the restrictions

$3x + 6y \le 1,$

$5x + 2y \le 1,$

$x \ge 0,\ \ y \ge 0$

Solution of LPP:

Introduce two slack variables s_1, s_2 . Then the problem is transformed into the following one:

$$\text{Maximize } \frac{1}{V} = x + y + 0.s_1 + 0.s_2$$

subject to the constraints

$$3x + 6y + 1.s_1 + 0.s_2 = 1,$$
$$5x + 2y + 0.s_1 + 1.s_2 = 1,$$
$$x \ge 0,\ y \ge 0,\ s_1 \ge 0,\ s_2 \ge 0$$

Let us note that the above equations can be written in the form of a single matrix equation as

A X = B

where A = $\begin{bmatrix} 3 & 6 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ s_1 \\ s_2 \end{bmatrix}$, B = $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The entries in B are referred to as the b – values. Initially, the basic variables are s_1, s_2. We have the following simplex tableau:

	x	y	s_1	s_2	b – value
s_1 - row	3	6	1	0	1
s_2 - row	5	2	0	1	1
Objective function row	-1	-1	0	0	0

Consider the negative elements in the objective function row. They are –1, -1. The absolute values are 1, 1. There is a tie between these coefficients. To resolve the tie, we select the variable *x*. We take the new basic variable as *x*. Consider the ratio of b-value to *x*-value. We have the following ratios:

s_1 - row : $\frac{1}{3}$

s_2 - row : $\frac{1}{5}$

Minimum of $\{\frac{1}{3}, \frac{1}{5}\} = \frac{1}{5}$.

Hence select s_2 as the leaving variable. Thus the pivotal element is 5. We obtain the following tableau at the end of Iteration No. 1.

	x	y	s_1	s_2	b – value
s_1 - row	0	$\frac{24}{5}$	1	$-\frac{3}{5}$	$\frac{2}{5}$
x - row	1	$\frac{2}{5}$	0	$\frac{1}{5}$	$\frac{1}{5}$
Objective function row	0	$-\frac{3}{5}$	0	$\frac{1}{5}$	$\frac{1}{5}$

Now, the negative element in the objective function row is $-\frac{3}{5}$. This corresponds to y. We take the new basic variable as y. Consider the ratio of b-value to y-value. We have the following ratios:

s_1 - row : $\left(\frac{2}{5}\right) \Big/ \left(\frac{24}{5}\right) = \frac{1}{12}$

x - row : $\left(\frac{1}{5}\right) \Big/ \left(\frac{2}{5}\right) = \frac{1}{2}$

Minimum of $\left\{\frac{1}{12}, \frac{1}{2}\right\} = \frac{1}{12}$

Hence select s_1 as the leaving variable. The pivotal element is $\frac{24}{5}$. We get the following tableau at the end of Iteration No. 2.

	x	y	s_1	s_2	b – value
y - row	0	1	$\frac{5}{24}$	$-\frac{1}{8}$	$\frac{1}{12}$
x - row	1	0	$-\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{6}$
Objective function row	0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$

Since both x and y have been made basic variables, we have reached the stopping condition.

The optimum value of $\frac{1}{V}$ is $\frac{1}{4}$. This is provided by x = $\frac{1}{6}$ and y = $\frac{1}{12}$. Thus the optimum value of the game is obtained as V = 4. Using the relations $\frac{r}{V} = x, \frac{s}{V} = y$, we obtain $r = \frac{4}{6} = \frac{2}{3}$ and $s = \frac{4}{12} = \frac{1}{3}$.

Problem 2:

Solve the following game:

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

Solution:

The game has no saddle point. It is a mixed game. Let the probability that B will use his first strategy be r. Let the probability that B will use his second strategy be s. Let V be the value of the game.

When A follows his first strategy:

The expected payoff to A (i.e., the expected loss to B) = 2 r +5 s.

The pay-off to A cannot exceed V. So we have

$$2\,r + 5\,s \le V \qquad \text{(I)}$$

When A follows his second strategy:

The expected pay-off to A (i.e., expected loss to B) = 4 r + s.

The pay-off to A cannot exceed V. Hence we obtain the condition

$$4\,r + s \le V \qquad \text{(II)}$$

From (I) and (II) we have

$$2\frac{r}{V} + 5\frac{s}{V} \le 1$$

$$and \;\; 4\frac{r}{V} + \frac{s}{V} \le 1$$

Substitute

$$\frac{r}{V} = x \; and \; \frac{s}{V} = y.$$

Thus we have

$$2x + 5y \le 1$$

$$and \;\; 4x + y \le 1$$

where r and s are connected by the relation

$$r + s = 1.$$

$$\text{i.e., } \frac{r}{V} + \frac{s}{V} = \frac{1}{V}$$

$$\text{i.e., } x + y = \frac{1}{V}$$

The objective of B is to minimise V. i.e., He will try to maximise $\frac{1}{V}$.

Thus we are led to the following linear programming problem:

$$\text{Maximize } \frac{1}{V} = x + y$$

subject to the constraints

$$2x + 5y \leq 1,$$
$$4x + \ \ y \leq 1,$$
$$x \geq 0,\ y \geq 0.$$

To solve this linear programming problem, we use simplex method as detailed below.

Introduce two slack variables s_1, s_2. Then the problem is transformed into the following one:

$$\text{Maximize } \frac{1}{V} = x + y + 0.s_1 + 0.s_2$$

subject to the constraints

$$2x + 5y + 1.s_1 + 0.s_2 = 1,$$
$$4x + \ \ y + 0.s_1 + 1.s_2 = 1,$$
$$x \geq 0,\ y \geq 0,\ s_1 \geq 0,\ s_2 \geq 0$$

We have the following simplex tableau:

	x	y	s_1	s_2	b – value
s_1 - row	2	5	1	0	1
s_2 - row	4	1	0	1	1
Objective function row	-1	-1	0	0	0

Consider the negative elements in the objective function row. They are –1, -1. The absolute value are 1, 1. There is a tie between these coefficients. To resolve the tie, we select the variable *x*. We take the new basic variable as *x*. Consider the ratio of b-value to *x*-value. We have the following ratios:

s_1 - row $\quad : \frac{1}{2}$

s_2 - row : $\frac{1}{4}$

Minimum of $\{\frac{1}{2}, \frac{1}{4}\} = \frac{1}{4}$

Hence select s_2 as the leaving variable. Thus the pivotal element is 4. We obtain the following tableau at the end of Iteration No. 1.

	x	y	s_1	s_2	b – value
s_1 - row	0	$\frac{9}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$
x - row	1	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$
Objective function row	0	$-\frac{3}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$

Now, the negative element in the objective function row is $-\frac{3}{4}$. This corresponds to y. We take the new basic variable as y. Consider the ratio of b-value to y-value. We have the following ratios:

s_1 - row : $\left(\frac{1}{2}\right) \Big/ \left(\frac{9}{2}\right) = \frac{1}{9}$

x - row : $\left(\frac{1}{4}\right) \Big/ \left(\frac{1}{4}\right) = 1$

Minimum of $\left\{\frac{1}{9}, 1\right\} = \frac{1}{9}$

Hence select s_1 as the leaving variable. The pivotal element is $\frac{9}{2}$. We get the following tableau at the end of Iteration No. 2.

	x	y	s_1	s_2	b – value
y - row	0	1	$\frac{2}{9}$	$-\frac{1}{9}$	$\frac{1}{9}$

x - row	1	0	$-\frac{1}{18}$	$\frac{5}{18}$	$\frac{2}{9}$
Objective function row	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$

Since both x and y have been made basic variables, we have reached the stopping condition.

The optimum value of $\frac{1}{V}$ is $\frac{1}{3}$.

This is provided by x = $\frac{2}{9}$ and y = $\frac{1}{9}$. Thus the optimum value of the game is got as V = 3.

Using the relations $\frac{r}{V} = x$, $\frac{s}{V} = y$, we obtain $r = \frac{6}{9} = \frac{2}{3}$ and $s = \frac{3}{9} = \frac{1}{3}$.

Problem 3:

Solve the following game by simplex method for LPP:

Player B

Player A $\begin{bmatrix} -48 & 2 \\ 6 & -4 \end{bmatrix}$

Solution :

Row minima : I row : -48

II row : -4

Maximum of {-48, -4} = -4

Column maxima: I column : 6

II column : 2

Minimum of (6, 2} = 2

So, Maximum of {Row minima} $\neq$ Minimum of {Column maxima}.

Therefore the given game has no saddle point. It is a mixed game. Let us convert the given game into a LPP.

Problem formulation:

Let V denote the value of the game. Let the probability that the player B will use his first strategy be r and second strategy be s. Let V denote the value of the game.

When A follows his first strategy:

The expected payoff to A (i.e., the expected loss to B) = - 48 r + 2 s.

This pay-off cannot exceed V. So we have